

# **Homogeneous Coordinates**

**CSCI 4229/5229  
Computer Graphics  
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# Basic 2D Transformations

Translation:

$$x' = x + \delta x$$

$$y' = y + \delta y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Scaling:

$$x' = S_x x$$

$$y' = S_y y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

$$x' = \cos \theta x - \sin \theta y$$

$$y' = \sin \theta x + \cos \theta y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

# Compound Transformations

Translation	$p' = p + d$
Scaling	$p' = Sp$
Rotation	$p' = Rp$

## Sequence of transformations

$$\begin{aligned} p' &= S_3 R_3 (S_2 (R_2 R_1 S_1 (p + d_1) + d_2) + d_3) \\ &= M_3 (M_2 (M_1 (p + d_1) + d_2) + d_3) \end{aligned}$$

# Homogeneous Coordinates

- Define

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \\ Z/W \end{pmatrix}$$

# 2D Homogeneous Transformations

Translation:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta x \\ 0 & 1 & \delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} x + w\delta x \\ y + w\delta y \\ w \end{pmatrix} \equiv \begin{pmatrix} x/w + \delta x \\ y/w + \delta y \\ 1 \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} S_x x \\ S_y y \\ w \end{pmatrix} \equiv \begin{pmatrix} S_x x/w \\ S_y y/w \\ 1 \end{pmatrix}$$

Rotation

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \cos \theta x - \sin \theta y \\ \sin \theta x + \cos \theta y \\ w \end{pmatrix} \\ \equiv \begin{pmatrix} [\cos \theta x - \sin \theta y]/w \\ [\sin \theta x + \cos \theta y]/w \\ 1 \end{pmatrix}$$

# Compound Homogeneous Transformations

$$\text{Translation} \quad p' = Dp$$

$$\text{Scaling} \quad p' = Sp$$

$$\text{Rotation} \quad p' = Rp$$

Sequence of transformations

$$\begin{aligned} p' &= S_3 R_3 D_3 S_2 D_2 R_2 R_1 S_1 D_1 p \\ &= Mp \end{aligned}$$

# 3D Homogeneous Translation and Scaling

Translation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta x \\ 0 & 1 & 0 & \delta y \\ 0 & 0 & 1 & \delta z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

# 3D Homogeneous Rotation

Rotation around the  $z$  axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the  $x$  axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the  $y$  axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

# General Homogeneous Rotation

Rotation around a unit vector  $(X, Y, Z)$ :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 + (1 - \cos \theta)(X^2 - 1) & -Z \sin \theta + (1 - \cos \theta)XY & Y \sin \theta + (1 - \cos \theta)XZ & 0 \\ Z \sin \theta + (1 - \cos \theta)XY & 1 + (1 - \cos \theta)(Y^2 - 1) & -X \sin \theta + (1 - \cos \theta)YZ & 0 \\ -Y \sin \theta + (1 - \cos \theta)XZ & X \sin \theta + (1 - \cos \theta)YZ & 1 + (1 - \cos \theta)(Z^2 - 1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotate  $x$  axis to unit vector  $(X_0, Y_0, Z_0)$  and  $y$  axis to unit vector  $(X_1, Y_1, Z_1)$ :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0Z_1 - Y_1Z_0 & 0 \\ Y_0 & Y_1 & Z_0X_1 - Z_1X_0 & 0 \\ Z_0 & Z_1 & X_0Y_1 - X_1Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

# Vector Rotation by Construction

Rotate  $x$  axis to unit vector  $(X_0, Y_0, Z_0)$  and  $y$  axis to unit vector  $(X_1, Y_1, Z_1)$ :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0Z_1 - Y_1Z_0 & 0 \\ Y_0 & Y_1 & Z_0X_1 - Z_1X_0 & 0 \\ Z_0 & Z_1 & X_0Y_1 - X_1Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

First column by construction

$$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ w \end{pmatrix} = \begin{pmatrix} X_0 & * & * & 0 \\ Y_0 & * & * & 0 \\ Z_0 & * & * & 0 \\ 0 & * & * & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ w \end{pmatrix}$$

Second column by construction

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ w \end{pmatrix} = \begin{pmatrix} * & X_1 & * & 0 \\ * & Y_1 & * & 0 \\ * & Z_1 & * & 0 \\ * & 0 & * & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ w \end{pmatrix}$$

Rigid transformation requires third column to be the cross product.

# 3D Homogeneous Transformation

Transformation matrices will always be of the form

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} r_{xx} & r_{yx} & r_{zx} & t_x \\ r_{xy} & r_{yy} & r_{zy} & t_y \\ r_{xz} & r_{yz} & r_{zz} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Note that translation, scaling and rotation or any combination of these three never changes the value of  $w$ 
  - There are transformations that do modify  $w$

# Order of calls in OpenGL

- What is the resulting  $x' = M_1 M_2 M_3 x$  for
  - `glTranslate(1000,2000,3000);`
  - `glRotate(90,0,0,1);`
  - `glScale(10,20,30);`
  - .....
  - `glVertex(1,0,0);`
- Answer:  $x' = (1000,2010,3000)$

# First Try

- Start with  $(1,0,0)$
- `glTranslate(1000,2000,3000);`  
 $(1,0,0) + (1000,2000,3000) =$   
 $(1001,2000,3000)$
- `glRotate(90,0,0,1);`  
 $(1001,2000,3000) \rightarrow (-2000,1001,3000)$
- `glScale(10,20,30);`  
 $(-2000,1001,3000) \rightarrow (-20000,20020,90000)$
- **WRONG:**
  - Correct answer:  $x' = (1000,2010,3000)$

# Second Try

- Start with (1,0,0)
- `glScale(10,20,30);`  
 $(1,0,0) \rightarrow (10,0,0)$
- `glRotate(90,0,0,1);`  
 $(10,0,0) \rightarrow (0,10,0)$
- `glTranslate(1000,2000,3000);`  
 $(0,10,0) + (1000,2000,3000) =$   
 $(1000,2010,3000)$
- Killer fact: OpenGL ***RIGHT*** multiplies, so read from `glVertex` **backwards**

# OpenGL Transformation Matrices

- You can set the matrices explicitly in OpenGL
  - `glLoadMatrixd(double mat[16])`
  - `glMultMatrixd(double mat[16])`
- Setting the matrix may be convenient when you calculate vectors instead of angles
- Killer fact:
  - The matrix is stored in COLUMN major order

# OpenGL 4 Transformations

- OpenGL 1 & 2 provides transformations as part of the API
- OpenGL 3 & 4 provides transformations only in the compatibility profile
  - Wants the user to generate transformations by other means
  - Provides matrix & vector operations, but user must set transformation matrices

# Useful Vector Results

- **$\mathbf{c} = \mathbf{a} \times \mathbf{b}$** 
  - **$\mathbf{c}$**  is perpendicular to the plane formed by  **$\mathbf{a}$**  and  **$\mathbf{b}$**
  - **$\mathbf{c}$**  is zero when  **$\mathbf{a}$**  and  **$\mathbf{b}$**  are parallel
- **$\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a})$** 
  - **$\mathbf{c}$**  is perpendicular to  **$\mathbf{a}$**
  - **$\mathbf{c}$**  is in the plane formed by  **$\mathbf{a}$**  and  **$\mathbf{b}$**
  - **$\mathbf{c}$**  is zero when  **$\mathbf{a}$**  and  **$\mathbf{b}$**  are parallel