

# **Parametric Curves**

**CSCI 4229/5229  
Computer Graphics  
Fall 2023**

# Parametric Curves

- $x(t) = p_x(t), y(t) = p_y(t), z(t) = p_z(t), w(t) = p_w(t)$
- Often  $p(t)$  is a polynomial
- Generally  $t$  in  $[0,1]$
- Avoids problems such as lines parallel to axes
- Works for any number of dimensions
- Can be used to generate vertexes, colors, texture coordinates, normals, etc.

# Bernstein Polynomials

- Bernstein Polynomials

- $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$

- Sums to one  $\sum_{i=0}^n B_i^n(t) = 1$

- Cubic Bernstein Polynomials

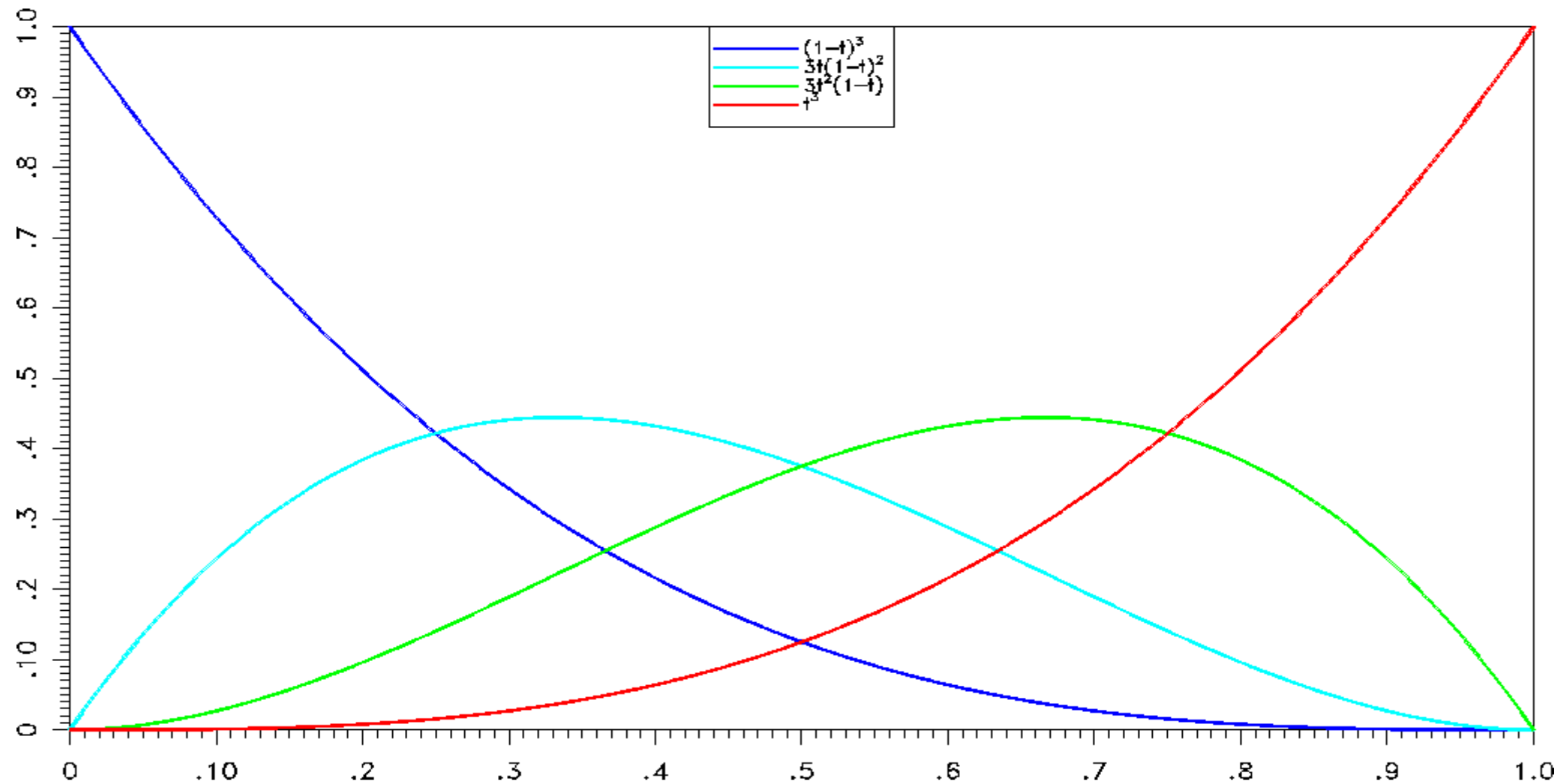
- $B_0^3(t) = (1-t)^3$

- $B_1^3(t) = 3t(1-t)^2$

- $B_2^3(t) = 3t^2(1-t)$

- $B_3^3(t) = t^3$

# Cubic Bernstein Polynomials



# Bézier Curves

- $C_n(t) = \sum_{i=0}^n B_i^n(t) P_i, \quad t \in [0,1]$
- $P_i$  are points in 2D, 3D or 4D
- Convex linear combination of points  $P_i$ 
  - Entire curve is in convex hull of points ( $\sum_{i=0}^n B_i^n(t)=1$ )
  - $B_0^n(0) = 1$ , so starts at  $P_0$
  - $B_n^n(1) = 1$ , so ends at  $P_n$
  - Tangential to  $P_0-P_1$  and  $P_n-P_{n-1}$
- Curve is smooth and differentiable

# Curves in OpenGL

- One-dimensional Evaluators
- Can be used to generate vertexes, normals, colors and textures
- Curve defined analytically using Bezier curves
- Evaluated at discrete points and rendered using straight line segments

# Curves in OpenGL

- `glEnable()`
  - Enables types of data to generate
- `glMap1d()`
  - Defines control points and domain
- `glEvalCoord1d()`
  - Generates a data point
- `glMapGrid1d()` & `glEvalMesh1()`
  - Generates a series of data points
- Deprecated in OpenGL (do this manually)

# ***glMap1d(type,Umin,Umax,stride,order,points)***

- *type* of data to generate
  - GL\_MAP1\_VERTEX\_3[4]
  - GL\_MAP1\_NORMAL
  - GL\_MAP1\_COLOR\_4
  - GL\_MAP1\_TEXTURE\_COORD\_[1-4]
- *Umin&Umax* are limits of parameter (often 0&1)
- *stride* is the number of coordinates in data (3 or 4)
- *order* is the order of the curve (4=cubic)
- *points* is the array of data points
- **Remember to also call glEnable()**



# glEvalCoord1d(u)

- Generate one vertex for each glMap1d() type currently active (e.g. texture, normal, vertex)
- To generate the whole curve, call glEvalCoord1d() once for each vertex
- Exercise entire parameter space
  - u from Umin to Umax (0 to 1)

# Generating a complete curve

- `glMapGrid1d(N , U1 , U2)`
- `glEvalMesh1(mode , N1 , N2)`
- This is equivalent to

```
glBegin(mode);
for (i=N1;i<=N2;i++)
    glEvalCoord1(U1 + i*(U2-U1)/N);
glEnd();
```

# Interpolation with Bézier Curves

- We have 4 points we want the curve to pass through  $P_0, P_1, P_2$  &  $P_3$
- What should control points  $R_0, R_1, R_2$  &  $R_3$  be?

$$P_0 = R_0 B_0^3(0) + R_1 B_1^3(0) + R_2 B_2^3(0) + R_3 B_3^3(0)$$

$$P_1 = R_0 B_0^3\left(\frac{1}{3}\right) + R_1 B_1^3\left(\frac{1}{3}\right) + R_2 B_2^3\left(\frac{1}{3}\right) + R_3 B_3^3\left(\frac{1}{3}\right)$$

$$P_2 = R_0 B_0^3\left(\frac{2}{3}\right) + R_1 B_1^3\left(\frac{2}{3}\right) + R_2 B_2^3\left(\frac{2}{3}\right) + R_3 B_3^3\left(\frac{2}{3}\right)$$

$$P_3 = R_0 B_0^3(1) + R_1 B_1^3(1) + R_2 B_2^3(1) + R_3 B_3^3(1)$$

# Relationship between $P$ and $R$

$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} B_0^3(0) & B_1^3(0) & B_2^3(0) & B_3^3(0) \\ B_0^3(\frac{1}{3}) & B_1^3(\frac{1}{3}) & B_2^3(\frac{1}{3}) & B_3^3(\frac{1}{3}) \\ B_0^3(\frac{2}{3}) & B_1^3(\frac{2}{3}) & B_2^3(\frac{2}{3}) & B_3^3(\frac{2}{3}) \\ B_0^3(1) & B_1^3(1) & B_2^3(1) & B_3^3(1) \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix}$$

$$\begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} B_0^3(0) & B_1^3(0) & B_2^3(0) & B_3^3(0) \\ B_0^3(\frac{1}{3}) & B_1^3(\frac{1}{3}) & B_2^3(\frac{1}{3}) & B_3^3(\frac{1}{3}) \\ B_0^3(\frac{2}{3}) & B_1^3(\frac{2}{3}) & B_2^3(\frac{2}{3}) & B_3^3(\frac{2}{3}) \\ B_0^3(1) & B_1^3(1) & B_2^3(1) & B_3^3(1) \end{pmatrix}^{-1} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

# Bézier Interpolation Matrix

- Selected so  $t=1/3$  and  $t=2/3$  maps to  $P_1$  &  $P_2$
- Local function (depends only on  $P_0, P_1, P_2, P_3$ )

$$\begin{array}{l}
 B_0^3(t) = (1-t)^3 \\
 B_1^3(t) = 3t(1-t)^2 \\
 B_2^3(t) = 3t^2(1-t) \\
 B_3^3(t) = t^3
 \end{array}
 \quad
 \begin{array}{l}
 B_0^3(0) = 1 \\
 B_1^3(0) = 0 \\
 B_2^3(0) = 0 \\
 B_3^3(0) = 0
 \end{array}
 \quad
 \begin{array}{l}
 B_0^3(\frac{1}{3}) = \frac{8}{27} \\
 B_1^3(\frac{1}{3}) = \frac{4}{9} \\
 B_2^3(\frac{1}{3}) = \frac{2}{9} \\
 B_3^3(\frac{1}{3}) = \frac{1}{27}
 \end{array}
 \quad
 \begin{array}{l}
 B_0^3(\frac{2}{3}) = \frac{1}{27} \\
 B_1^3(\frac{2}{3}) = \frac{2}{9} \\
 B_2^3(\frac{2}{3}) = \frac{4}{9} \\
 B_3^3(\frac{2}{3}) = \frac{8}{27}
 \end{array}
 \quad
 \begin{array}{l}
 B_0^3(1) = 0 \\
 B_1^3(1) = 0 \\
 B_2^3(1) = 0 \\
 B_3^3(1) = 1
 \end{array}$$

$$\begin{pmatrix} 1 & \frac{8}{27} & \frac{1}{27} & 0 \\ 0 & \frac{4}{9} & \frac{2}{9} & 0 \\ 0 & \frac{2}{9} & \frac{4}{9} & 0 \\ 0 & \frac{1}{27} & \frac{8}{27} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Extending to More Points

- Two curves  $P_0, P_1, P_2, P_3$  and  $P_4, P_5, P_6, P_7$
- Bézier curves pass through  $P_0 \& P_3$  and  $P_4 \& P_7$ , so the curve will be continuous if  $P_3 = P_4$
- Bézier curves are tangential to  $P_1 - P_0$  &  $P_2 - P_3$  and  $P_5 - P_4$  &  $P_6 - P_7$ , so the curve will be smooth if  $P_3 = P_4$  and  $P_2 - P_3$  and  $P_5 - P_4$ , therefore  $P_5 = 2P_3 - P_2$

# Splines

- Traditionally a long, thin, flexible piece of wood or metal used to describe a smooth curve
  - Used in building boats, airplanes, etc.
- Held down by *ducks* or *whales*
- Mathematical equivalents
  - Natural Cubic Spline
  - Weights called *knots*
  - Piecewise polynomial



# Parametric Splines

- Three or four splines, one for each component
- Parameter  $t$  reach integer values at each knot
  - Cardinal spline
- Natural Cubic Spline
- Clamped Cubic Spline
- Quadratic Spline
- Hermite Spline



# Cardinal Cubic Spline

$$S_j(t) = \frac{(j+1-t)^3 - (j+1-t)}{6}G_j + \frac{(t-j)^3 - (t-j)}{6}G_{j+1} + (j+1-t)F_j + (t-j)F_{j+1}$$

$$S'_j(t) = \frac{1 - 3(j+1-t)^2}{6}G_j + \frac{3(t-j)^2 - 1}{6}G_{j+1} + F_{j+1} - F_j$$

$$S''_j(t) = (j+1-t)G_j + (t-j)G_{j+1}$$

$$S_j(j) = F_j$$

$$S_{j-1}(j) = F_j$$

$$S''_j(j) = G_j$$

$$S''_{j-1}(j) = G_j$$

$$S'_j(j) = -\frac{1}{3}G_j - \frac{1}{6}G_{j+1} + F_{j+1} - F_j$$

$$S'_{j-1}(j) = \frac{1}{6}G_{j-1} - \frac{1}{3}G_j + F_j - F_{j-1}$$

$$S'_{j-1}(j) = S'_j(j)$$

$$\frac{1}{6}G_{j-1} + \frac{2}{3}G_j + \frac{1}{6}G_{j+1} = F_{j-1} - 2F_j + F_{j+1}$$

# Cardinal Cubic Spline

- Requires solution of tri-diagonal matrix
- Global (all knots impact everywhere)

$$\frac{1}{6}G_{j-1} + \frac{2}{3}G_j + \frac{1}{6}G_{j+1} = F_{j-1} - 2F_j + F_{j+1}$$

$$\begin{pmatrix} 1 & & & & & & \\ \frac{1}{6} & & & & & & \\ & \frac{2}{3} & & & & & \\ & \frac{1}{6} & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \frac{1}{6} & & \\ & & & & \frac{2}{3} & & \\ & & & & \frac{1}{6} & & \\ & & & & & 1 & \end{pmatrix} \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \vdots \\ G_{n-2} \\ G_{n-1} \end{pmatrix} = \begin{pmatrix} F_0'' \\ F_0 - 2F_1 + F_2 \\ F_1 - 2F_2 + F_3 \\ \vdots \\ F_{n-3} - 2F_{n-2} + F_{n-1} \\ F_{n-1}'' \end{pmatrix}$$