Ray - Object Intersection

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Ray

A ray is defined as all points $\mathbf{p}$ such that

$$\mathbf{p} = \mathbf{o} + t\mathbf{d}, \qquad \forall t \geq 0. \tag{1}$$

Note that the ray starts at a point $\mathbf{o}$ and goes in one direction $\mathbf{d}$. $\mathbf{d}$ may not be zero or the direction of the ray will be undefined.

Ray-Plane intersection

A plane can be defined using a point in the plane $\mathbf{a}$ and a normal to the plane $\mathbf{n}$. Therefore all points $\mathbf{p}$ in the plane can be defined as

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0. \tag{2}$$

The point at which the ray intesects the plane can be found by subtitution of Eq. 1 into Eq. 2 so that

$$(\mathbf{o} + t\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0. \tag{3}$$

Solving for t yields

$$t = \frac{(\mathbf{a} - \mathbf{o}) \cdot \mathbf{n}}{\mathbf{d} \cdot \mathbf{n}}. \tag{4}$$

Eq. 4 always has a unique solution unless $\mathbf{d} \cdot \mathbf{n} = 0$, that is the ray is parallel to the plane. When $\mathbf{d} \cdot \mathbf{n}$ the ray never intersects the plane unless $\mathbf{o}$ is in the plane, in which case the entire ray is in the plane. Neither $\mathbf{d}$ or $\mathbf{n}$ can be zero or the ray or plane would be undefined.

The solution to Eq. 4 can be either positive or negative. If $t < 0$, the ray is away from the plane and will never intersect it. If $t > 0$ the ray is towards the plane and will eventually intersect it. If $t = 0$ the ray is starting in the plane.

One $t > 0$ is found, the point at which the ray intersects the plane is readily found by solving Eq. 1.

Ray-Sphere Intersection

A sphere can be defined using a center point $\mathbf{c}$ and radius r, so that all points on the sphere is defined by

$$(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) = r^2. \tag{5}$$

The point at which the ray intesects the plane can be found by subtitution of Eq. 1 into Eq. 5 so that

$$(\mathbf{o} + t\mathbf{d} - \mathbf{c}) \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{c}) = r^2. \tag{6}$$

Grouping by t yields

$$[\mathbf{d} \cdot \mathbf{d}]t^2 + 2[(\mathbf{o} - \mathbf{c}) \cdot \mathbf{d}]t + [(\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - r^2] = 0. \tag{7}$$

This is a quadratic equation in t which can be solved as

$$t = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{8}$$
$$\alpha = \mathbf{d} \cdot \mathbf{d} \tag{9}$$
$$\beta = 2(\mathbf{o} - \mathbf{c}) \cdot \mathbf{d} \tag{10}$$
$$\gamma = (\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - r^2 \tag{11}$$

When $\mathbf{d}$ is a unit vector, then $\alpha = \mathbf{d} \cdot \mathbf{d} = 1$, so that Eq. 8 simplifies to

$$t = \mu \pm \sqrt{\delta}, \tag{12}$$
$$\mathbf{h} = \mathbf{c} - \mathbf{o}, \tag{13}$$
$$\delta = \mu^2 - \mathbf{h} \cdot \mathbf{h} + r^2, \tag{14}$$
$$\mu = \mathbf{h} \cdot \mathbf{d}. \tag{15}$$

When $\delta < 0$, then t has no real solution. This means that the ray passes further than r from the center of the sphere, and never intersects it.

When $\delta = 0$, then t has exactly one solution. This means that the ray passes the sphere tangentially. If, however, the value of $t < 0$, then the ray is away from the sphere, and does not intersect it.

When $\delta > 0$, then t has two real solutions. If both values of t are negative, then the ray is away from the sphere. If one value of t is positive and the other negative, the ray starts inside the sphere, and exits at the positive value. If both values of t are positive, then the smaller t value is where the ray inters the sphere, and the larger t value is wher the ray exists the sphere.

Ray Intersections with Implicit Surfaces

For any surface that can be implicitly defined as $f(\mathbf{p}) = 0$, we can find the intersection of a ray with that surface by solving $f(\mathbf{o}+t\mathbf{d}) = 0$. In the case of the plane and the sphere, this can be done quite simply as shown above. For more complex surfaces, this may not be possible.

For example, a solution exists for the intersection of a ray and a torus. A solution can also be found for a disk, a cylinder and a cone, but in the case of these objects the solution consists of two steps: Finding a solution to the intersection of the ray with an infinitely large version of the object, and then the finite sized object.

The key to using such an analytic description of an object in ray tracing is that the solution must be fast as these object-ray intersection calculations is the most time consuming part of the procedure.

Ray-Triangle Intersection

A triangle can be define using the three vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}, \tag{16}$$

$$1 = \alpha + \beta + \gamma, \tag{17}$$

$$0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1, \quad 0 \leq \gamma \leq 1. \tag{18}$$

Combining Eqs. 1, 12 and 13 using $\alpha = 1 - \beta - \gamma$ yields

$$\mathbf{o} + t\mathbf{d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}), \tag{19}$$

which can be rearranged as

$$\beta(\mathbf{a} - \mathbf{b}) + \gamma(\mathbf{a} - \mathbf{c}). + t\mathbf{d} = \mathbf{a} - \mathbf{o}. \tag{20}$$

Eq. 20 contains three unknowns, t, β and γ. However, since the variables are in three dimensions, the equation can be rewritten as

$$\begin{pmatrix} \mathbf{a}_x - \mathbf{b}_x & \mathbf{a}_x - \mathbf{c}_x & \mathbf{d}_x \\ \mathbf{a}_y - \mathbf{b}_y & \mathbf{a}_y - \mathbf{c}_y & \mathbf{d}_y \\ \mathbf{a}_z - \mathbf{b}_z & \mathbf{a}_z - \mathbf{c}_z & \mathbf{d}_z \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ t \end{pmatrix} = \begin{pmatrix} \mathbf{a}_x - \mathbf{o}_x \\ \mathbf{a}_y - \mathbf{o}_y \\ \mathbf{a}_z - \mathbf{o}_z \end{pmatrix} \tag{21}$$

The solution to Eq. 21 can be found using Cramer's rule which uses determinants as

$$D = \begin{vmatrix} \mathbf{a}_x - \mathbf{b}_x & \mathbf{a}_x - \mathbf{c}_x & \mathbf{d}_x \\ \mathbf{a}_y - \mathbf{b}_y & \mathbf{a}_y - \mathbf{c}_y & \mathbf{d}_y \\ \mathbf{a}_z - \mathbf{b}_z & \mathbf{a}_z - \mathbf{c}_z & \mathbf{d}_z \end{vmatrix} \tag{22}$$

$$\beta = \begin{vmatrix} \mathbf{a}_x - \mathbf{o}_x & \mathbf{a}_x - \mathbf{c}_x & \mathbf{d}_x \\ \mathbf{a}_y - \mathbf{o}_y & \mathbf{a}_y - \mathbf{c}_y & \mathbf{d}_y \\ \mathbf{a}_z - \mathbf{o}_z & \mathbf{a}_z - \mathbf{c}_z & \mathbf{d}_z \end{vmatrix} /D \tag{23}$$

$$\gamma = \begin{vmatrix} \mathbf{a}_x - \mathbf{b}_x & \mathbf{a}_x - \mathbf{o}_x & \mathbf{d}_x \\ \mathbf{a}_y - \mathbf{b}_y & \mathbf{a}_y - \mathbf{o}_y & \mathbf{d}_y \\ \mathbf{a}_z - \mathbf{b}_z & \mathbf{a}_z - \mathbf{o}_z & \mathbf{d}_z \end{vmatrix} /D \tag{24}$$

$$t = \begin{vmatrix} \mathbf{a}_x - \mathbf{b}_x & \mathbf{a}_x - \mathbf{c}_x & \mathbf{a}_x - \mathbf{o}_x \\ \mathbf{a}_y - \mathbf{b}_y & \mathbf{a}_y - \mathbf{c}_y & \mathbf{a}_y - \mathbf{o}_y \\ \mathbf{a}_z - \mathbf{b}_z & \mathbf{a}_z - \mathbf{c}_z & \mathbf{a}_z - \mathbf{o}_z \end{vmatrix} /D \tag{25}$$

For the ray to hit the triangle, $t > 0$, $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$. When $D = 0$ the ray is parallel to the plane defined by the triangle, and no intersection occurs, or the ray may be in the plane and an infinite number of intersections may occur.

Bounding sphere

For complex objects, computing the ray-object intersection may be expensive. An optimization can be to compute a bounding sphere, that is a sphere containing the entire object. If the ray does not intersect the bounding sphere, then it cannot intersect the object. This provides a quick test that can be used to determine if it is necessary to perform the more expensive ray-object intersection calculations. Even more sophisticated techniques have been developed because this is such a critical step.