Mandelbulb

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This description is based on the tutorial by Iñigo Quilez
http://www.iquilezles.org/www/articles/mandelbulb/mandelbulb.htm
which contains many images and technical information about the Mandebulb.

The Mandelbrot set is the oldest and most famous example of a self-similar fractal set. This means that as you look at increasingly smaller areas of the set, the patterns seen at a larger scale repeats, and the complexity remains at smaller scales.

The Mandelbrot set is defined as the set of points $c \in M$ such that the sequence

$$Z_{k+1} = Z_k^2 + c, \quad Z_0 = 0,$$

converges, where Z is a complex number.

This can be generalized to other exponents as

$$Z_{k+1} = Z_k^n + c, \quad Z_0 = 0.$$

The Mandelbrot set is by definition two dimensional since it uses complex variables. To translate this to three dimensions, we use the White and Nylander spherical transformation of (r, θ, ϕ) to define

$$[x, y, z]^n = r^n[\cos(n\theta)\cos(n\phi)\ ,\ \sin(n\theta)cos(n\phi)\ ,\ -\sin(n\phi)]$$

where

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \\ \theta &= tan^{-1}\left(\frac{y}{x}\right), \\ \phi &= tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right). \end{aligned}$$

The resulting set for which this sequence converges is called the Mandelbulb.

Rendering the Mandelbrot is computationally intensive, but straight forward. Since the set is two dimensional, the sequence can be evaluated for each pixel on a plane, and the pixel colored accordingly.

However, in three dimensions, the problem becomes computationally intractible. It would be impractical to evaluate the sequence for each point in three dimensional space.

One possible solution is to use ray tracing. The concept is to find the point along the ray where the ray intersects the Mandelbulb. The brute force approach would be to evalute the sequence at different points along the ray and do a search to locate the edge of the Mandelbulb.

Fortunately, there is a better solution. Given a point c, the distance d to the Mandelbulb can be estimated as

$$d = \frac{G(c)}{|G'(c)|}$$

where $G(c)$ is Green's function, also known as the Hubbard-Douady potential, which is defined as

$$G(c) = \lim_{k\to\infty} \frac{1}{n^k} \ln |Z_k|$$

so that

$$|G'(c)| = \lim_{k\to\infty} \frac{1}{n^k} \frac{|Z'_k|}{Z_k}$$

and

$$d = \lim_{k\to\infty} \frac{|Z_k| \ln |Z_k|}{|Z'_k|}.$$

Since $Z_{k+1} = Z_k^n + c$, the derivative can be recursively defined as

$$Z'_{k+1} = nZ_k^{n-1} Z'_k + 1, \quad Z'_0 = 1.$$

Also note that $|Z_k| = r_k$.

In practice, the distance estimate is used iteratively, so it is not necessary to follow the sequence infinitely far.