Homogeneous Coordinates

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Computer Graphics
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Basic 2D Transformations

Translation:

$$x' = x + \delta x$$
$$y' = y + \delta y$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Scaling:

$$x' = S_x x$$
$$y' = S_y y$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

$$x' = \cos\theta x - \sin\theta y$$
$$y' = \sin\theta x + \cos\theta y$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Compound Transformations

Translation	$p' = p + d$
Scaling	$p' = Sp$
Rotation	$p' = Rp$

Sequence of transformations

$$
\begin{aligned}
p' &= S_3 R_3 (S_2 (R_2 R_1 S_1 (p + d_1) + d_2) + d_3) \\
&= M_3 (M_2 (M_1 (p + d_1) + d_2) + d_3)
\end{aligned}
$$

Homogeneous Coordinates

- Define

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \\ Z/W \end{pmatrix}$$

2D Homogeneous Transformations

Translation:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta x \\ 0 & 1 & \delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} x + w\delta x \\ y + w\delta y \\ w \end{pmatrix} \equiv \begin{pmatrix} x/w + \delta x \\ y/w + \delta y \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} S_x x \\ S_y y \\ w \end{pmatrix} \equiv \begin{pmatrix} S_x x/w \\ S_y y/w \end{pmatrix}$$

Rotation

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \cos\theta\ x - \sin\theta\ y \\ \sin\theta\ x + \cos\theta\ y \\ w \end{pmatrix}$$

$$\equiv \begin{pmatrix} [\cos\theta\ x - \sin\theta\ y]/w \\ [\sin\theta\ x + \cos\theta\ y]/w \end{pmatrix}$$

Compound Homogeneous Transformations

Translation	$p' = Dp$
Scaling	$p' = Sp$
Rotation	$p' = Rp$

Sequence of transformations

$$\begin{aligned} p' &= S_3 R_3 D_3 S_2 D_2 R_2 R_1 S_1 D_1 p \\ &= Mp \end{aligned}$$

3D Homogeneous Translation and Scaling

Translation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta x \\ 0 & 1 & 0 & \delta y \\ 0 & 0 & 1 & \delta z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

3D Homogeneous Rotation

Rotation around the z axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the x axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the y axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

General Homogeneous Rotation

Rotation around a unit vector (X, Y, Z):

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1+(1-\cos\theta)(X^2-1) & -Z\sin\theta+(1-\cos\theta)XY & Y\sin\theta+(1-\cos\theta)XZ & 0 \\ Z\sin\theta+(1-\cos\theta)XY & 1+(1-\cos\theta)(Y^2-1) & -X\sin\theta+(1-\cos\theta)YZ & 0 \\ -Y\sin\theta+(1-\cos\theta)XZ & X\sin\theta+(1-\cos\theta)YZ & 1+(1-\cos\theta)(Z^2-1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotate x axis to unit vector (X_0, Y_0, Z_0) and y axis to unit vector (X_1, Y_1, Z_1):

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0Z_1 - Y_1Z_0 & 0 \\ Y_0 & Y_1 & Z_0X_1 - Z_1X_0 & 0 \\ Z_0 & Z_1 & X_0Y_1 - X_1Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Vector Rotation by Construction

Rotate x axis to unit vector (X_0, Y_0, Z_0) and y axis to unit vector (X_1, Y_1, Z_1):

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0 Z_1 - Y_1 Z_0 & 0 \\ Y_0 & Y_1 & Z_0 X_1 - Z_1 X_0 & 0 \\ Z_0 & Z_1 & X_0 Y_1 - X_1 Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

First column by construction

$$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ w \end{pmatrix} = \begin{pmatrix} X_0 & * & * & 0 \\ Y_0 & * & * & 0 \\ Z_0 & * & * & 0 \\ 0 & * & * & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ w \end{pmatrix}$$

Second column by construction

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ w \end{pmatrix} = \begin{pmatrix} * & X_1 & * & 0 \\ * & Y_1 & * & 0 \\ * & Z_1 & * & 0 \\ * & 0 & * & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ w \end{pmatrix}$$

Rigid transformation requires third column to be the cross product.

3D Homogeneous Transformation

Transformation matrices will always be of the form

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} r_{xx} & r_{yx} & r_{zx} & t_x \\ r_{xy} & r_{yy} & r_{zy} & t_y \\ r_{xz} & r_{yz} & r_{zz} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Note that translation, scaling and rotation or any combination of these three never changes the value of *w*
  - There are transformations that do modify *w*

Order of calls in OpenGL

- What is the resulting $x' = M_1 M_2 M_3 x$ for
  - glTranslate(1000,2000,3000);
  - glRotate(90,0,0,1);
  - glScale(10,20,30);
  - .....
  - glVertex(1,0,0);
- Answer: $x' = (1000,2010,3000)$

First Try

- Start with (1,0,0)
- glTranslate(1000,2000,3000);

  (1,0,0) + (1000,2000,3000) =
    (1001,2000,3000)

- glRotate(90,0,0,1);

  (1001,2000,3000) -> (-2000,1001,3000)

- glScale(10,20,30);

  (-2000,1001,3000) -> (-20000,20020,90000)

- WRONG:
  - Correct answer: x' = (1000,2010,3000)

Second Try

- Start with (1,0,0)
- glScale(10,20,30);

  (1,0,0) -> (10,0,0)

- glRotate(90,0,0,1);

  (10,0,0) -> (0,10,0)

- glTranslate(1000,2000,3000);

  (0,10,0) + (1000,2000,3000) =
  (1000,2010,3000)

- Killer fact: OpenGL ***RIGHT*** multiplies, so read from glVertex **backwards**

OpenGL Transformation Matrices

- You can set the matrices explicitly in OpenGL
  - glLoadMatrixd(double mat[16])
  - glMultMatrixd(double mat[16])
- Setting the matrix may be convenient when you calculate vectors instead of angles
- Killer fact:
  - The matrix is stored in COLUMN major order

OpenGL 4 Transformations

- OpenGL 1 & 2 provides transformations as part of the API
- OpenGL 3 & 4 provides transformations only in the compatibility profile
  - Wants the user to generate transformations by other means
  - Provides matrix & vector operations, but user must set transformation matrices

Useful Vector Results

- **c** = **a** x **b**
  - **c** is perpendicular to the plane formed by **a** and **b**
  - **c** is zero when **a** and **b** are parallel
- **c** = **a** x (**b** x **a**)
  - **c** is perpendicular to **a**
  - **c** is in the plane formed by **a** and **b**
  - **c** is zero when **a** and **b** are parallel