Parametric Curves

CSCI 4229/5229
Computer Graphics
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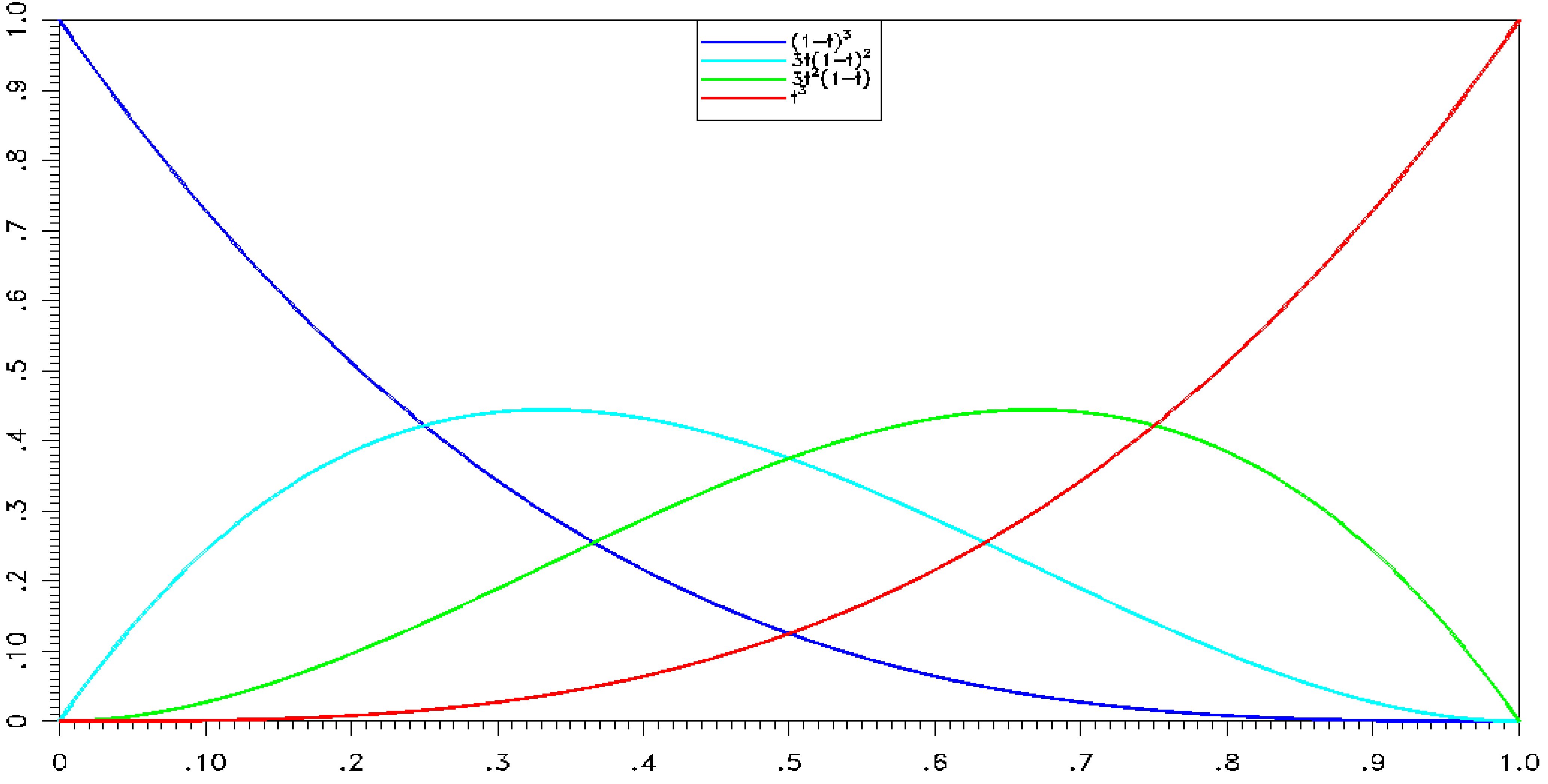
Parametric Curves

- $x(t) = p_x(t)$, $y(t) = p_y(t)$, $z(t) = p_z(t)$, $w(t) = p_w(t)$
- Often $p(t)$ is a polynomial
- Generally t in [0,1]
- Avoids problems such as lines parallel to axes
- Works for any number of dimensions
- Can be used to generate vertexes, colors, texture coordinates, normals, etc.

Bernstein Polynomials

- Bernstein Polynomials
  - $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$
- Sums to one $\sum_{i=0}^{n} B_i^n(t) = 1$
- Cubic Bernstein Polynomials
  - $B_0^3(t) = (1-t)^3$
  - $B_1^3(t) = 3t(1-t)^2$
  - $B_2^3(t) = 3t^2(1-t)$
  - $B_3^3(t) = t^3$

Cubic Bernstein Polynomials



Bézier Curves

- $C_n(t) = \sum_{i=0}^{n} B_i^n(t)\, P_i, \quad t \in [0,1]$
- P_i are points in 2D, 3D or 4D
- Convex linear combination of points P_i
  - Entire curve is in convex hull of points ($\sum_{i=0}^{n} B_i^n(t)=1$)
  - $B_0^n(0) = 1$, so starts at P_0
  - $B_n^n(1) = 1$, so ends at P_n
  - Tangential to P_0-P_1 and P_n-P_{n-1}
- Curve is smooth and differentiable

Curves in OpenGL

- One-dimensional Evaluators
- Can be used to generate vertexes, normals, colors and textures
- Curve defined analytically using Bezier curves
- Evaluated at discrete points and rendered using straight line segments

Curves in OpenGL

- glEnable()
  - Enables types of data to generate
- glMap1d()
  - Defines control points and domain
- glEvalCoord1d()
  - Generates a data point
- glMapGrid1d() & glEvalMesh1()
  - Generates a series of data points
- Deprecated in OpenGL (do this manually)

glMap1d(*type,Umin,Umax,stride,order,points*)

- *type* of data to generate
  - GL_MAP1_VERTEX_[34]
  - GL_MAP1_NORMAL
  - GL_MAP1_COLOR_4
  - GL_MAP1_TEXTURE_COORD_[1-4]
- *Umin*&*Umax* are limits of parameter(often0&1)
- *stride* is the number of coordinates in data (3 or 4)
- *order* is the order of the curve (4=cubic)
- *points* is the array of data points
- **Remember to also call glEnable()**

glEvalCoord1d(u)

- Generate one vertex for each glMap1d() type currently active (e.g. texture, normal, vertex)
- To generate the whole curve, call glEvalCoord1d() once for each vertex
- Exercise entire parameter space
  - u from Umin to Umax (0 to 1)

Generating a complete curve

- glMapGrid1d(N , U1 , U2)
- glEvalMesh1(mode , N1 , N2)
- This is equivalent to

```
glBegin(mode);
for (i=N1;i<=N2;i++)
    glEvalCoord1(U1 + i*(U2-U1)/N);
glEnd();
```

Interpolation with Bézier Curves

- We have 4 points we want the curve to pass through P_0, P_1, P_2 & P_3
- What should control points R_0, R_1, R_2 & R_3 be?

$$P_0 = R_0 B_0^3(0) + R_1 B_1^3(0) + R_2 B_2^3(0) + R_3 B_3^3(0)$$
$$P_1 = R_0 B_0^3(\tfrac{1}{3}) + R_1 B_1^3(\tfrac{1}{3}) + R_2 B_2^3(\tfrac{1}{3}) + R_3 B_3^3(\tfrac{1}{3})$$
$$P_2 = R_0 B_0^3(\tfrac{2}{3}) + R_1 B_1^3(\tfrac{2}{3}) + R_2 B_2^3(\tfrac{2}{3}) + R_3 B_3^3(\tfrac{2}{3})$$
$$P_3 = R_0 B_0^3(1) + R_1 B_1^3(1) + R_2 B_2^3(1) + R_3 B_3^3(1)$$

Relationship between P and R

$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} B_0^3(0) & B_1^3(0) & B_2^3(0) & B_3^3(0) \\ B_0^3(\frac{1}{3}) & B_1^3(\frac{1}{3}) & B_2^3(\frac{1}{3}) & B_3^3(\frac{1}{3}) \\ B_0^3(\frac{2}{3}) & B_1^3(\frac{2}{3}) & B_2^3(\frac{2}{3}) & B_3^3(\frac{2}{3}) \\ B_0^3(1) & B_1^3(1) & B_2^3(1) & B_3^3(1) \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix}$$

$$\begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} B_0^3(0) & B_1^3(0) & B_2^3(0) & B_3^3(0) \\ B_0^3(\frac{1}{3}) & B_1^3(\frac{1}{3}) & B_2^3(\frac{1}{3}) & B_3^3(\frac{1}{3}) \\ B_0^3(\frac{2}{3}) & B_1^3(\frac{2}{3}) & B_2^3(\frac{2}{3}) & B_3^3(\frac{2}{3}) \\ B_0^3(1) & B_1^3(1) & B_2^3(1) & B_3^3(1) \end{pmatrix}^{-1} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Bézier Interpolation Matrix

- Selected so t=1/3 and t=2/3 maps to P_1 & P_2
- Local function (depends only on P_0, P_1, P_2, P_3)

$$
\begin{aligned}
B_0^3(t) &= (1-t)^3 & B_0^3(0) &= 1 & B_0^3(\tfrac{1}{3}) &= \tfrac{8}{27} & B_0^3(\tfrac{2}{3}) &= \tfrac{1}{27} & B_0^3(1) &= 0 \\
B_1^3(t) &= 3t(1-t)^2 & B_1^3(0) &= 0 & B_1^3(\tfrac{1}{3}) &= \tfrac{4}{9} & B_1^3(\tfrac{2}{3}) &= \tfrac{2}{9} & B_1^3(1) &= 0 \\
B_2^3(t) &= 3t^2(1-t) & B_2^3(0) &= 0 & B_2^3(\tfrac{1}{3}) &= \tfrac{2}{9} & B_2^3(\tfrac{2}{3}) &= \tfrac{4}{9} & B_2^3(1) &= 0 \\
B_3^3(t) &= t^3 & B_3^3(0) &= 0 & B_3^3(\tfrac{1}{3}) &= \tfrac{1}{27} & B_3^3(\tfrac{2}{3}) &= \tfrac{8}{27} & B_3^3(1) &= 1
\end{aligned}
$$

$$
\begin{pmatrix} 1 & \frac{8}{27} & \frac{1}{27} & 0 \\ 0 & \frac{4}{9} & \frac{2}{9} & 0 \\ 0 & \frac{2}{9} & \frac{4}{9} & 0 \\ 0 & \frac{1}{27} & \frac{8}{27} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Extending to More Points

- Two curves P_0, P_1, P_2, P_3 and P_4, P_5, P_6, P_7
- Bézier curves pass through P_0&P_3 and P_4&P_7, so the curve will be continuous of $P_3 = P_4$
- Bézier curves are tangential to $P_1 - P_0$ & $P_2 - P_3$ and $P_5 - P_4$ & $P_6 - P_7$, so the curve will be smooth if $P_3 = P_4$ and $P_2 - P_3$ and $P_5 - P_4$, therefor $P_5 = 2P_3 - P_2$

Splines

- Traditionally a long, thin, flexible piece of wood or metal used to describe a smooth curve
  - Used in building boats, airplanes, etc.
- Held down by *ducks* or *whales*
- Mathematical equivalents
  - Natural Cubic Spline
  - Weights called *knots*
  - Piecewise polynomial

Parametric Splines

- Three or four splines, one for each component
- Parameter *t* reach integer values at each knot
  - Cardinal spline
- Natural Cubic Spline
- Clamped Cubic Spline
- Quadratic Spline
- Hermite Spline

Cardinal Cubic Spline

$$S_j(t) = \frac{(j+1-t)^3 - (j+1-t)}{6} G_j + \frac{(t-j)^3 - (t-j)}{6} G_{j+1} + (j+1-t)F_j + (t-j)F_{j+1}$$

$$S_j'(t) = \frac{1 - 3(j+1-t)^2}{6} G_j + \frac{3(t-j)^2 - 1}{6} G_{j+1} + F_{j+1} - F_j$$

$$S_j''(t) = (j+1-t)G_j + (t-j)G_{j+1}$$

$$\begin{aligned} S_j(j) &= F_j \\ S_{j-1}(j) &= F_j \\ S_j''(j) &= G_j \\ S_{j-1}''(j) &= G_j \\ S_j'(j) &= -\frac{1}{3}G_j - \frac{1}{6}G_{j+1} + F_{j+1} - F_j \\ S_{j-1}'(j) &= \frac{1}{6}G_{j-1} - \frac{1}{3}G_j + F_j - F_{j-1} \end{aligned}$$

$$S_{j-1}'(j) = S_j'(j)$$

$$\frac{1}{6}G_{j-1} + \frac{2}{3}G_j + \frac{1}{6}G_{j+1} = F_{j-1} - 2F_j + F_{j+1}$$

Cardinal Cubic Spline

- Requires solution of tri-diagonal matrix
- Global (all knots impact everywhere)

$$\frac{1}{6}G_{j-1} + \frac{2}{3}G_j + \frac{1}{6}G_{j+1} = F_{j-1} - 2F_j + F_{j+1}$$

$$\begin{pmatrix} 1 & & & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \vdots \\ G_{n-2} \\ G_{n-1} \end{pmatrix} = \begin{pmatrix} F_0'' \\ F_0 - 2F_1 + F_2 \\ F_1 - 2F_2 + F_3 \\ \vdots \\ F_{n-3} - 2F_{n-2} + F_{n-1} \\ F_{n-1}'' \end{pmatrix}$$