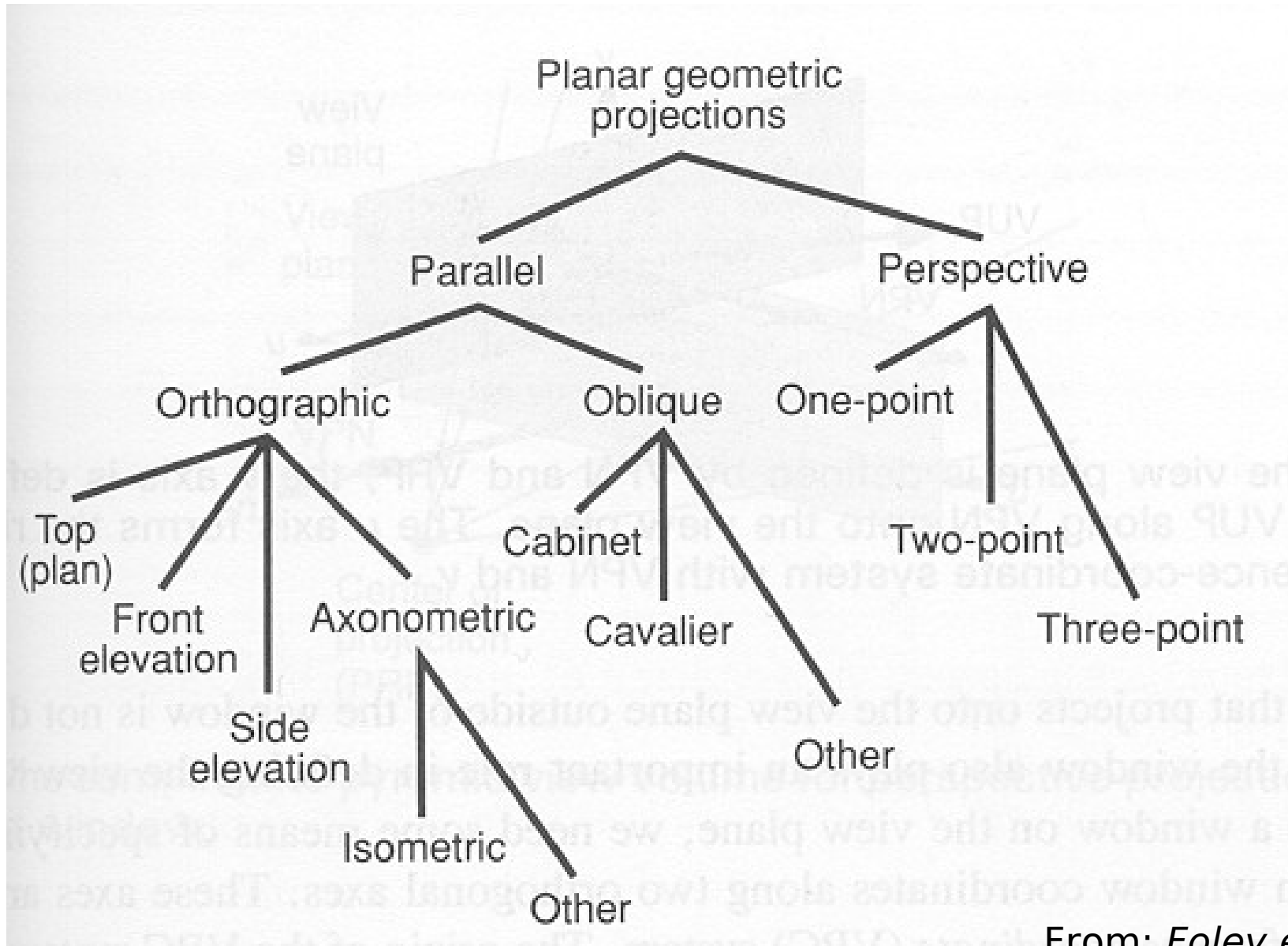# Drawing in 3D Projections

CSCI 4229/5229
Computer Graphics
Summer 2023

Types of Projections

- Parallel Projections
  - Orthogonal, isometric, ...
  - Size does not diminish with distance
- Perspective
  - Realistic based on an observer's point of view
  - Nearer bigger, farther smaller
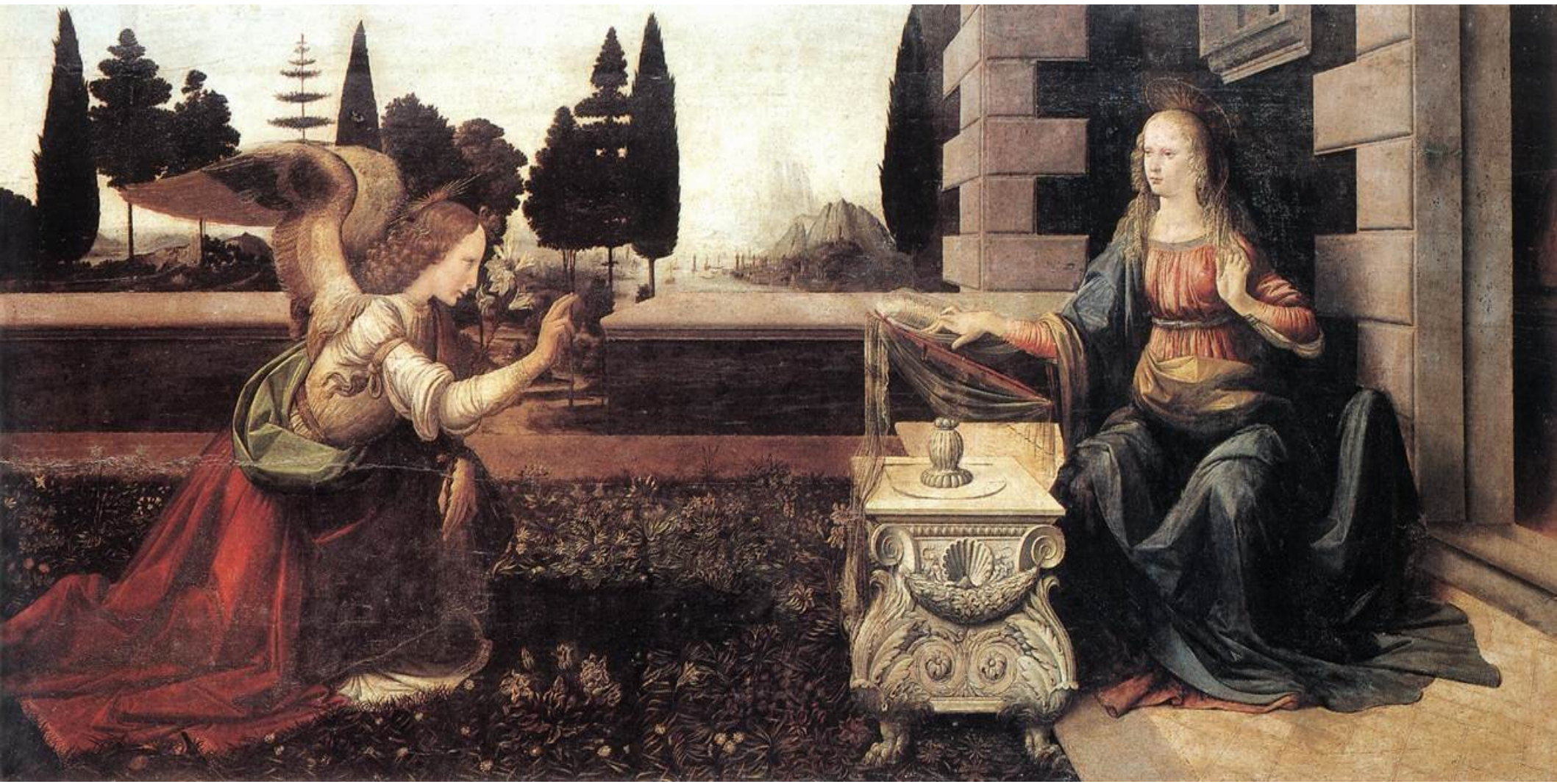 - One or more vanishing points
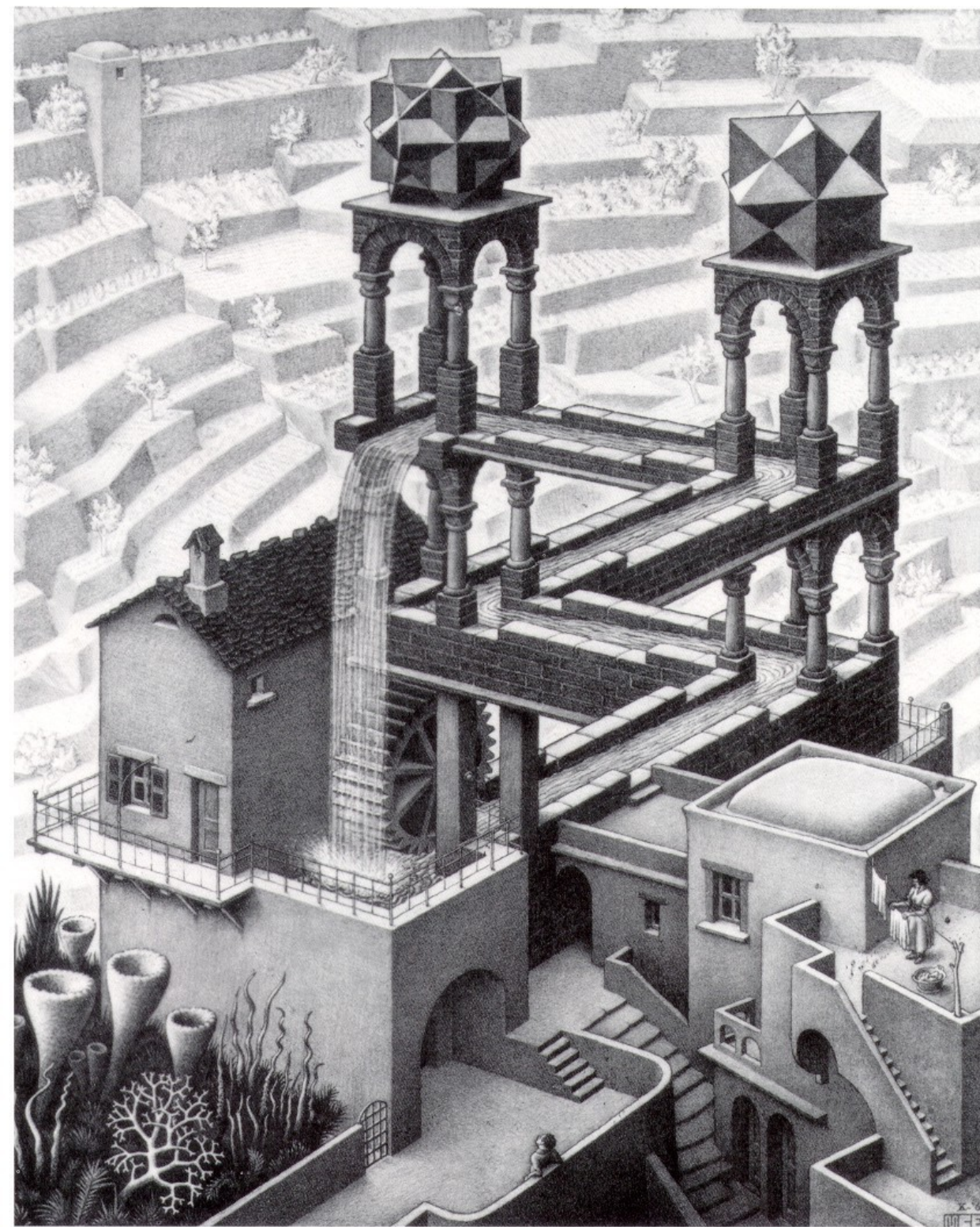
Taxonomy of Projections

From: *Foley et al*
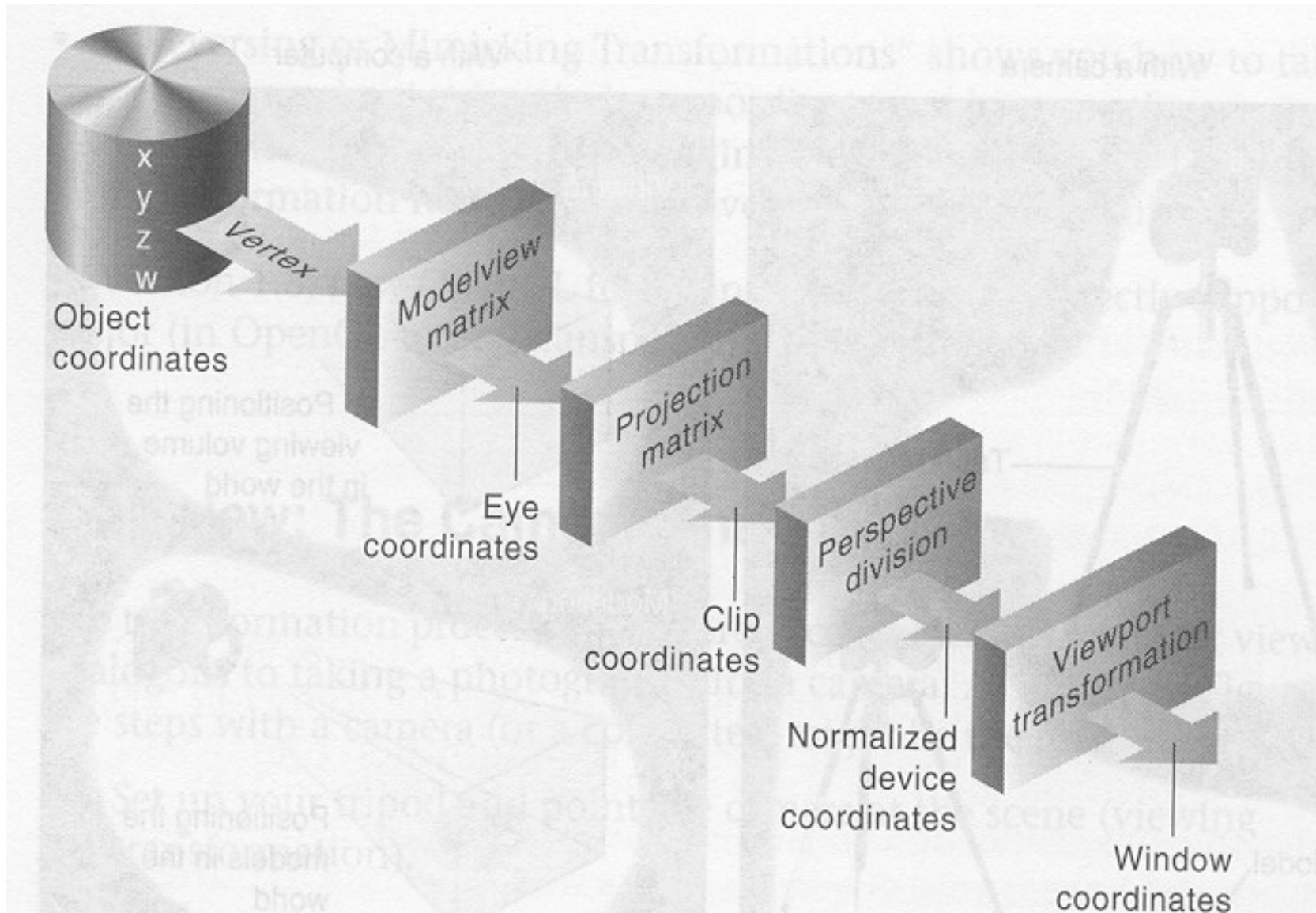
Egyptian Tomb Painting

Annunciation
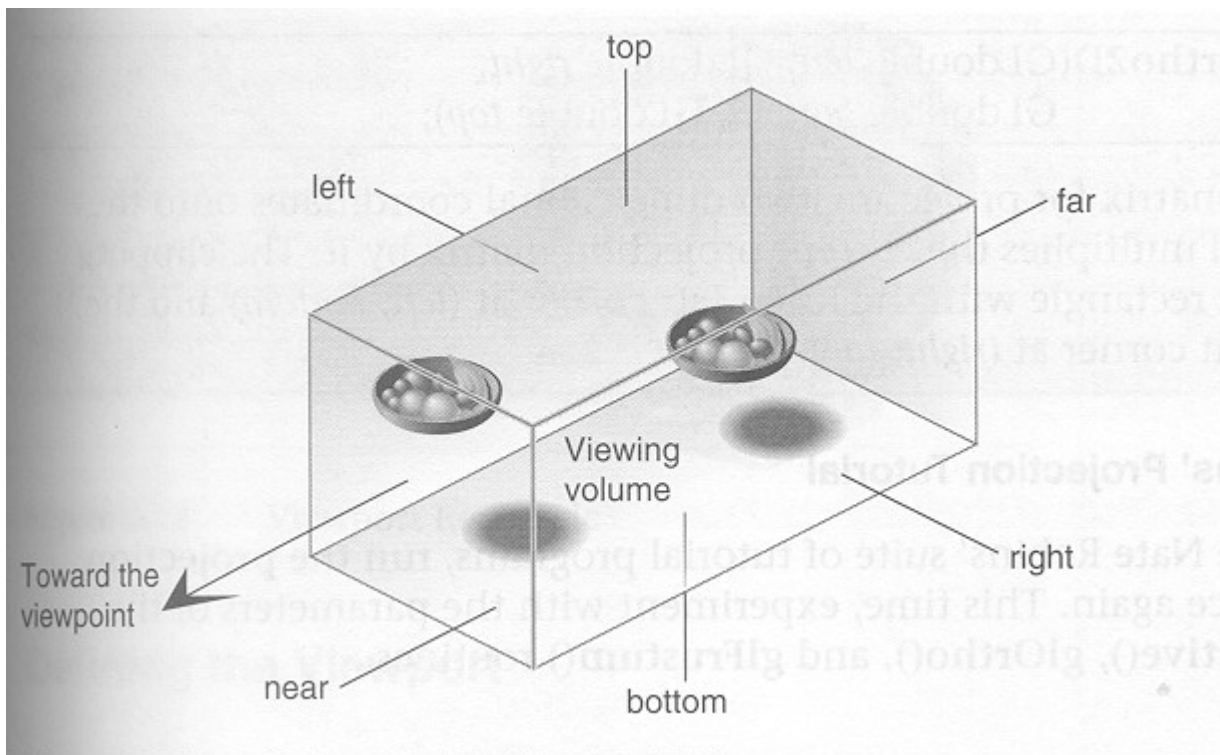# Leonardo da Vinci (1472)

Waterfall
M.C.
Escher
(1961)

OpenGL Transformation Pipeline

Parallel Projection

- Apply rotation matrix to map direction of projection to *Z* axis and up to *Y* axis

- Scale to canonical volume

From: *OpenGL Red Book*

glOrtho(x_{min}, x_{max}, y_{min}, y_{max}, z_{min}, z_{max})

glOrtho Projection Matrix

$$\begin{pmatrix} \frac{2}{x_{max}-x_{min}} & 0 & 0 & -\frac{x_{max}+x_{min}}{x_{max}-x_{min}} \\ 0 & \frac{2}{y_{max}-y_{min}} & 0 & -\frac{y_{max}+y_{min}}{y_{max}-y_{min}} \\ 0 & 0 & \frac{-2}{z_{max}-z_{min}} & \frac{z_{max}+z_{min}}{z_{max}-z_{min}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Perspective Transformation

- Apply rotation matrix to map eye position to center of scene to negative *Z* and up to *Y* axes

- Scale *(x,y)* inversely proportional to distance

- Scale to canonical volume
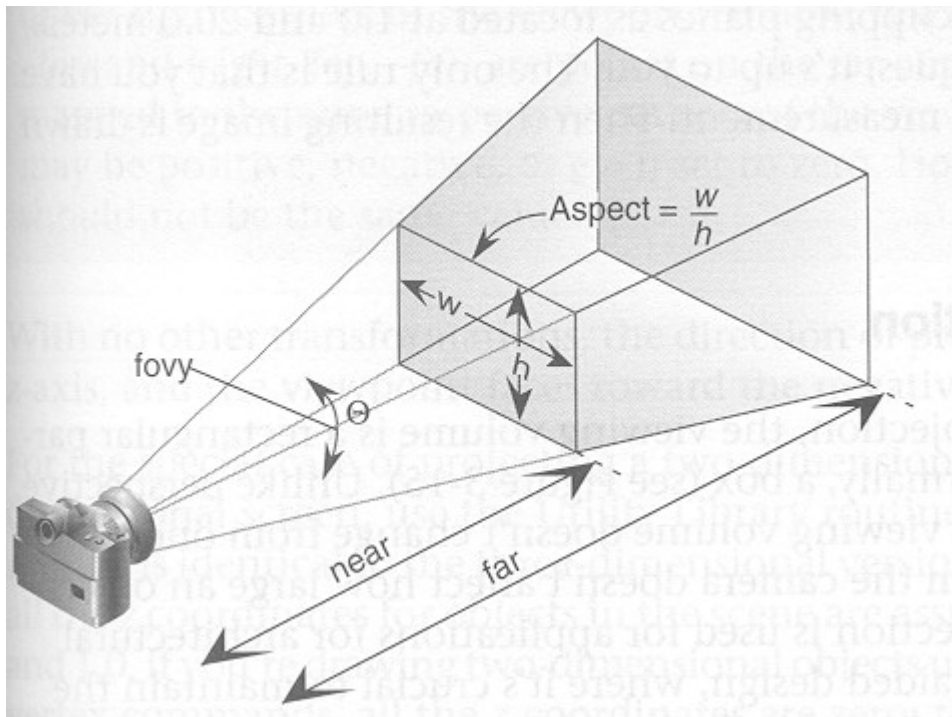
From: *OpenGL Red Book*

Perspective Transformation
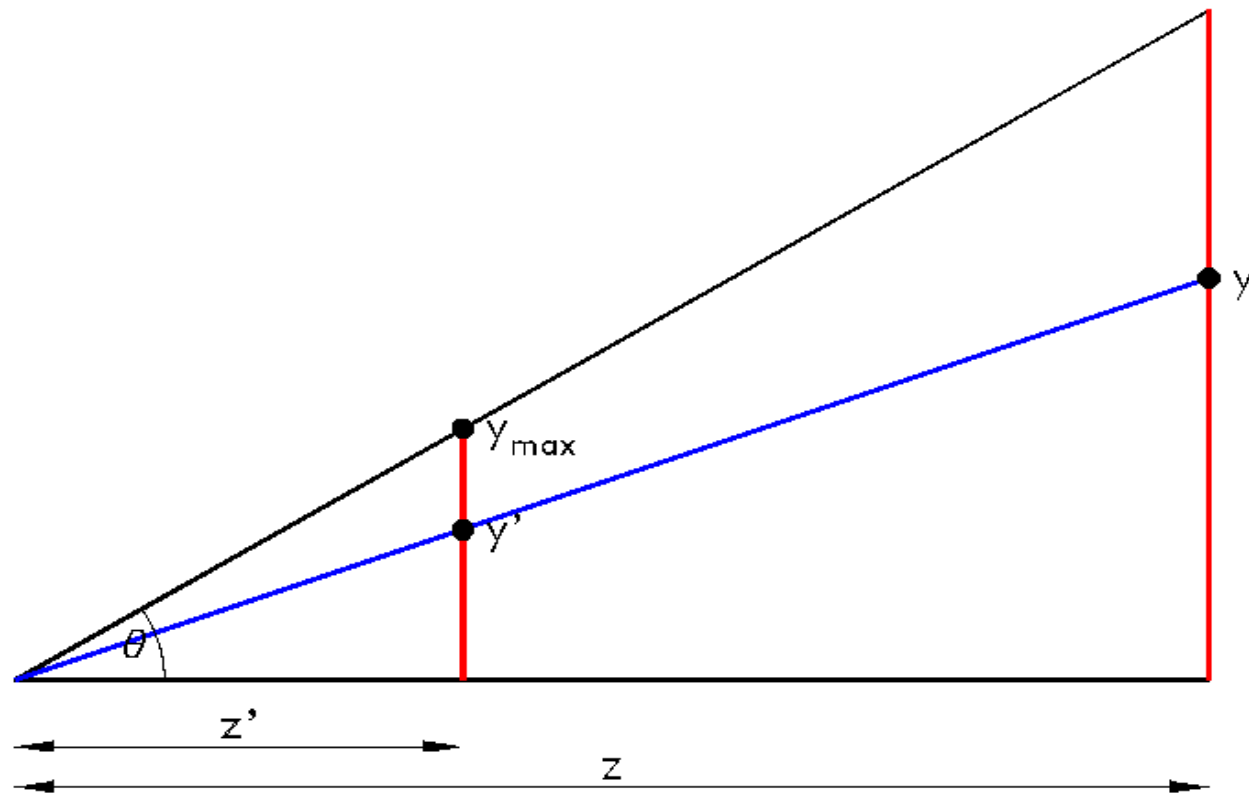
Similar triangles: $y'/z' = y/z$,   so $y' = y/z \; z'$

Let $y_{max} = 1$ (NDC), $\tan \theta = y_{max} / z'$, $z' = \cot \theta$

$$y' = \cot \theta \; y/z$$

Homogeneous Perspective Multiply

$$
\begin{pmatrix}
\frac{\cot\theta}{aspect} & 0 & 0 & 0 \\
0 & \cot\theta & 0 & 0 \\
0 & 0 & \frac{z_{far}+z_{near}}{z_{far}-z_{near}} & \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z \\ 1
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\frac{\cot\theta}{aspect}\, x \\
\cot\theta\, y \\
\frac{z_{far}+z_{near}}{z_{far}-z_{near}} z + \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\
-z
\end{pmatrix}
$$

$$
\equiv
\begin{pmatrix}
\frac{\cot\theta}{aspect}\frac{x}{z} \\
\cot\theta\frac{y}{z} \\
\frac{-2z_{far}z_{near}}{z(z_{far}-z_{near})} - \frac{z_{far}+z_{near}}{z_{far}-z_{near}}
\end{pmatrix}
$$

gluPerspective(fovy,aspect,Znear,Zfar)

- *fovy* is the angle in the up/down direction

- *aspect* is is the horizontal to vertical ratio

- *Znear* is the distance to the near clipping plane

  – Killer fact Znear > 0

- *Zfar* is the distance to the far clipping plane

  – *Zfar > Znear*

- *Zfar-Znear* determines *Z* resolution since the Z buffer has finite precision

  – $\log_2($ *Zfar/Znear* $)$ bits lost

gluPerspective(fovy,aspect,Znear,Zfar)

Let θ = fovy/2

gluPerspective Projection Matrix

$$
\begin{pmatrix}
\frac{\cot\theta}{aspect} & 0 & 0 & 0 \\
0 & \cot\theta & 0 & 0 \\
0 & 0 & -\frac{z_{far}+z_{near}}{z_{far}-z_{near}} & \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\
0 & 0 & -1 & 0
\end{pmatrix}
$$

gluLookAt(E_x, E_y, E_z , C_x, C_y, C_z , U_x, U_y, U_z)

- (E_x, E_y, E_z) is the eye position
- (C_x, C_y, C_z) is the position you look at
- (U_x, U_y, U_z) is the up direction
- *C-E* determines the distance in the *Z* direction
- The *Z* distance to each object (from E) determines the reduction in the *(x,y)* direction

gluLookAt(E_x, E_y, E_z , C_x, C_y, C_z , U_x, U_y, U_z)

Forward $\quad$ $\mathbf{F} = \mathbf{C} - \mathbf{E}$

Sideways $\quad$ $\mathbf{S} = \mathbf{F} \times \mathbf{U}$

Up $\quad$ $\mathbf{U} = \mathbf{S} \times \mathbf{F}$

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & U_x & -F_x & 0 \\ S_y & U_y & -F_y & 0 \\ S_z & U_z & -F_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -E_x \\ 0 & 1 & 0 & -E_y \\ 0 & 0 & 1 & -E_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} S_x & U_x & -F_x & -E_x S_x - E_y U_x + E_z F_x \\ S_y & U_y & -F_y & -E_x S_y - E_y U_y + E_z F_y \\ S_z & U_z & -F_z & -E_x S_z - E_y U_z + E_z F_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

($\mathbf{F}$ and $\mathbf{U}$ must be normalized)

First Person Navigation

- Decide where you are (E_x, E_y, E_z)

- Decide which way you are looking

  - $(C_x, C_y, C_z) = (E_x, E_y, E_z) + (d_x, d_y, d_z)$

- Decide up, e.g. (0,0,1)

- Move forward to new position

  - $(E_x, E_y, E_z) \mathrel{+}= dt*(d_x, d_y, d_z)$

- Turn left and right using angle

  - $(d_x, d_y, d_z) = (\cos\theta, \sin\theta, 0)$