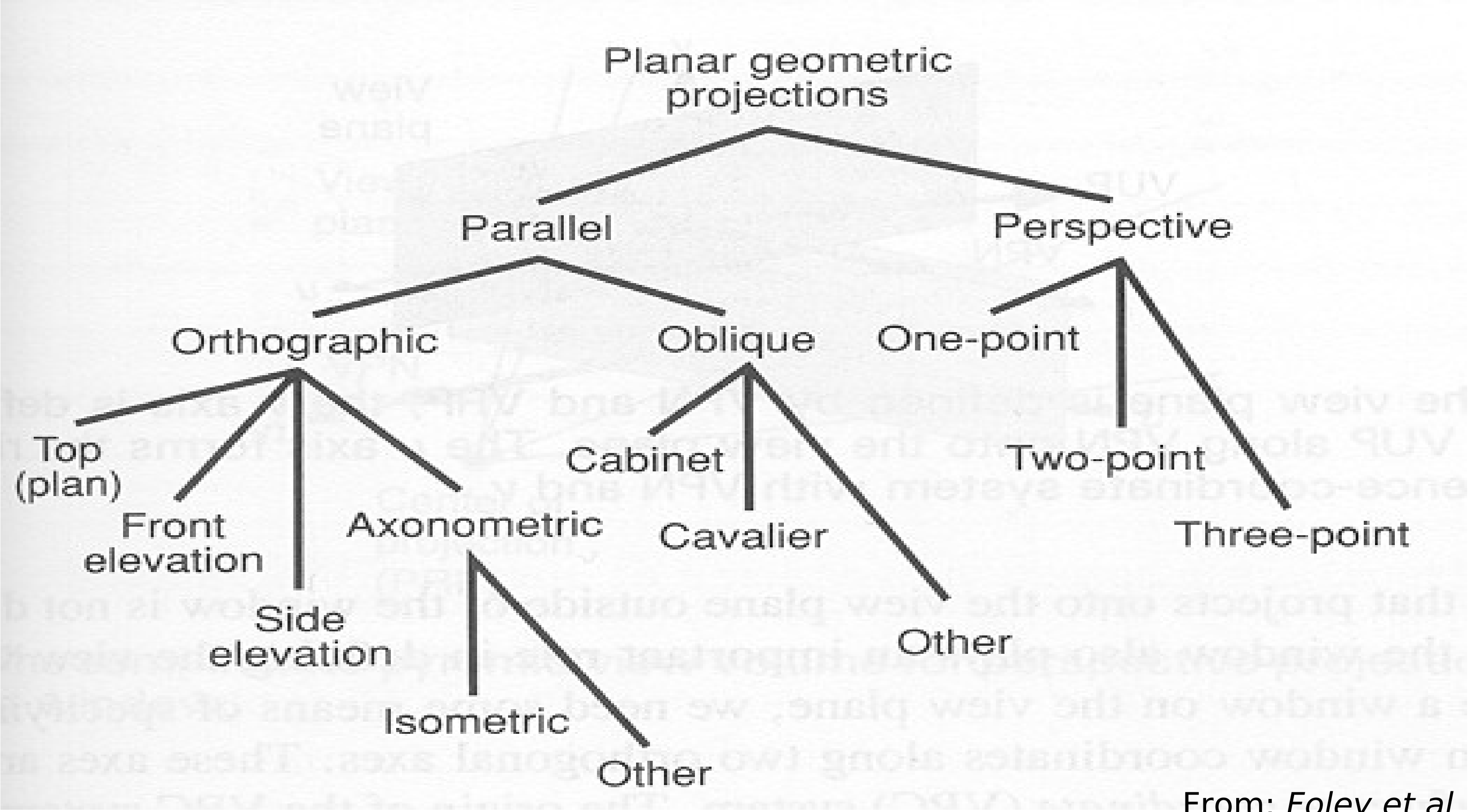
Drawing in 3D Projections

CSCI 4229/5229
Computer Graphics
Fall 2025

Types of Projections

- Parallel Projections
  - Orthogonal, isometric, ...
  - Size does not diminish with distance
- Perspective
  - Realistic based on an observer's point of view
  - Nearer bigger, farther smaller
  - One or more vanishing points

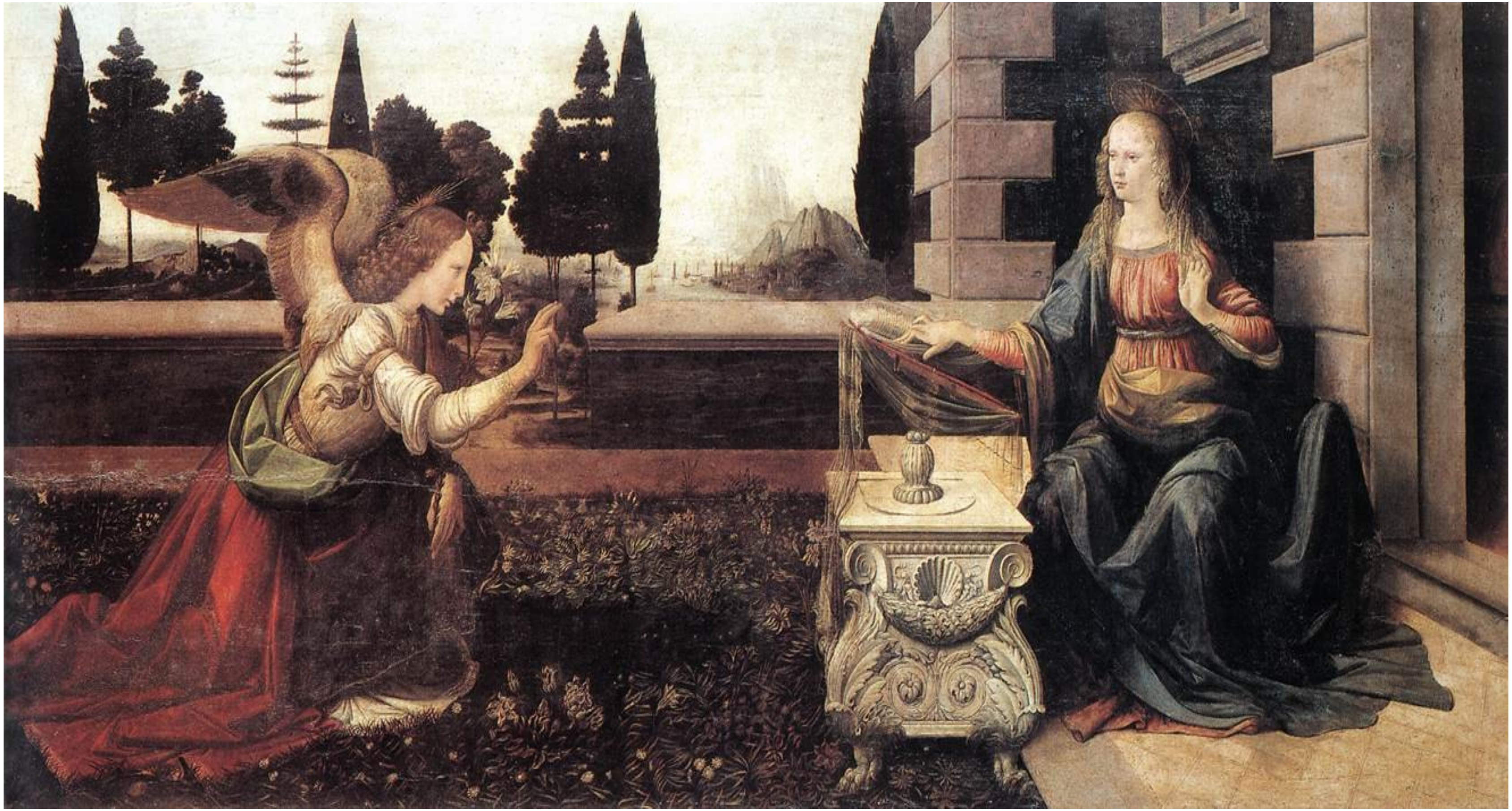
Taxonomy of Projections

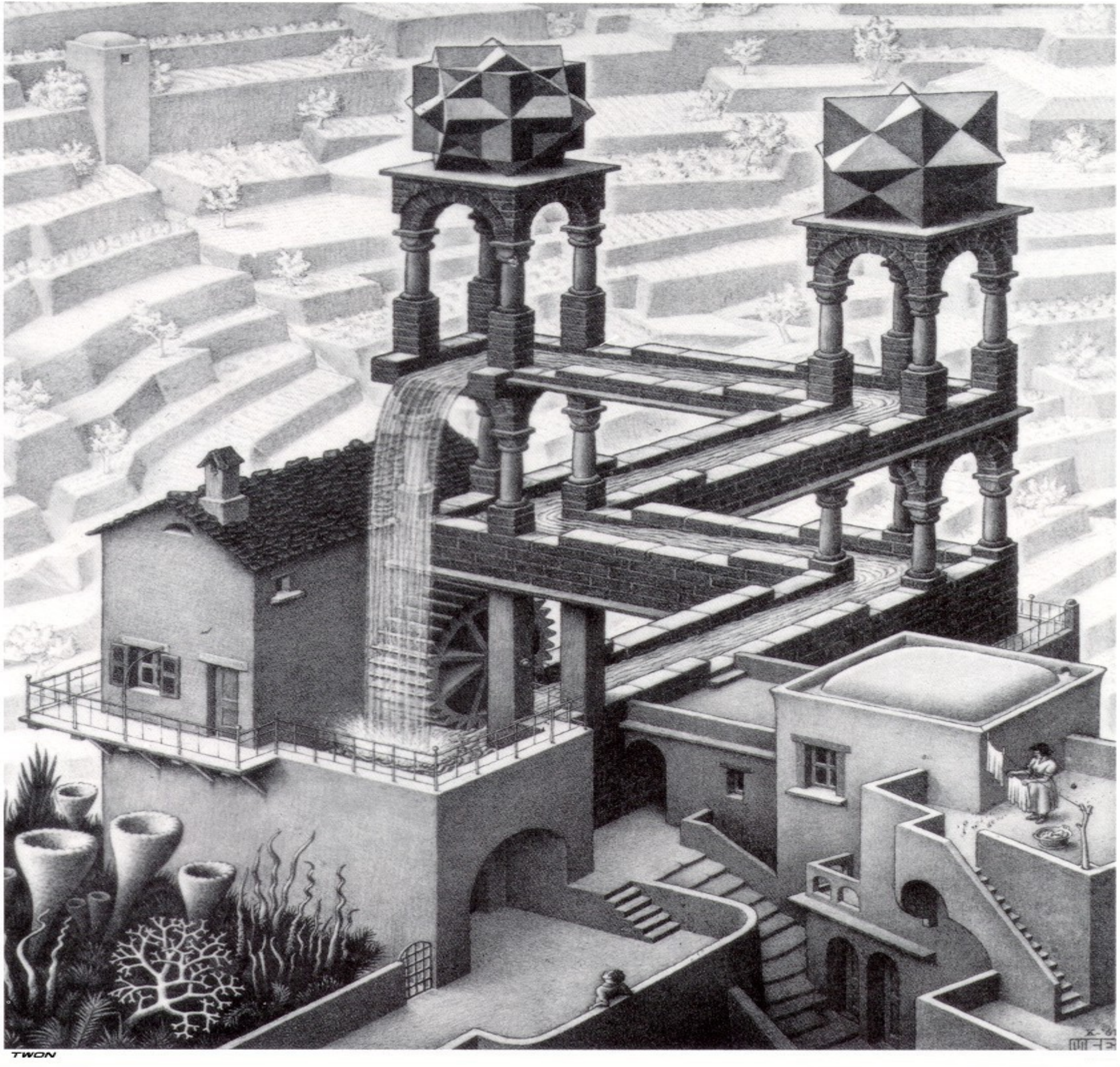


From: *Foley et al*

Papyrus of Ani

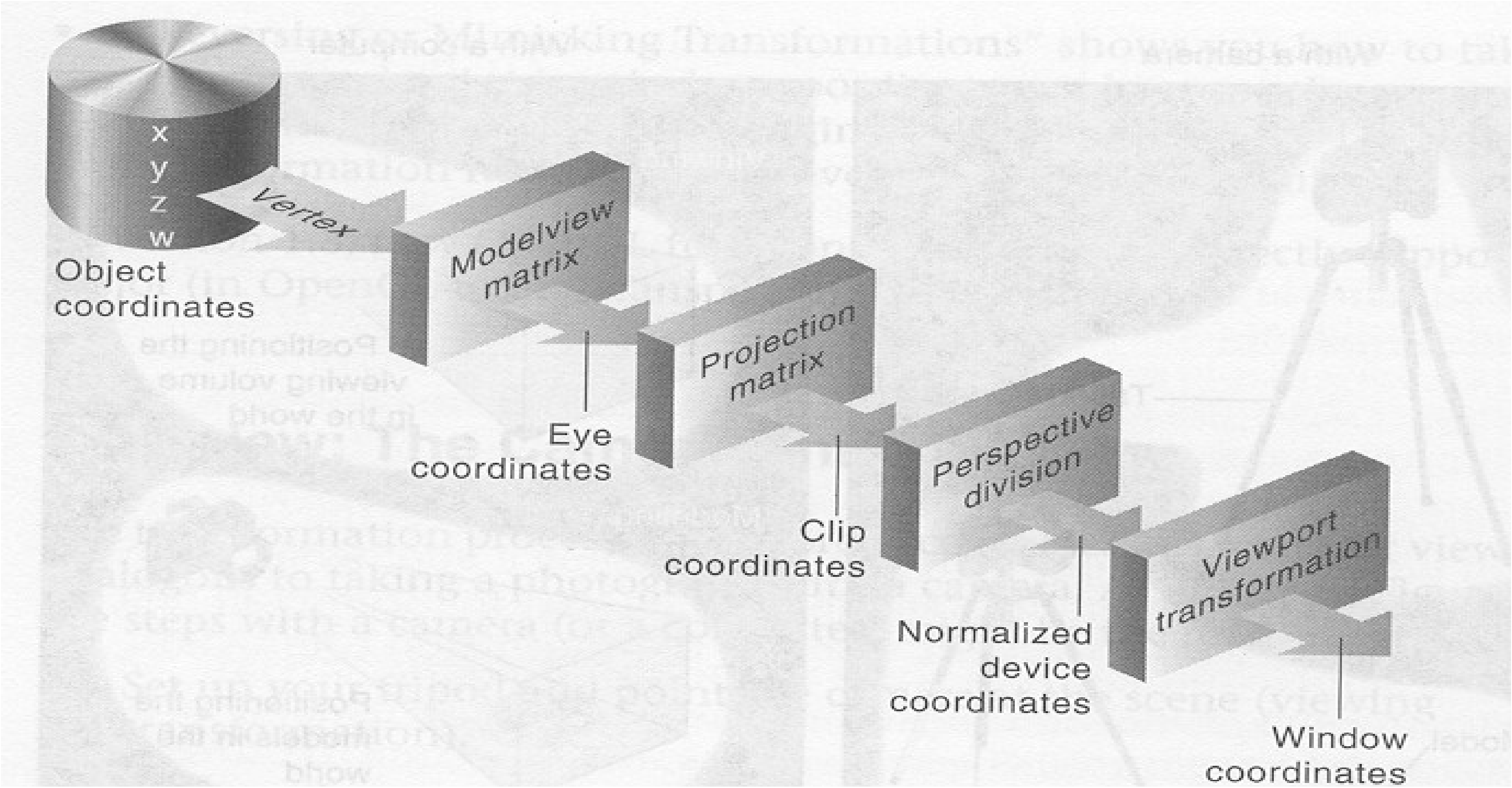
Annunciation
Leonardo da Vinci (1472)





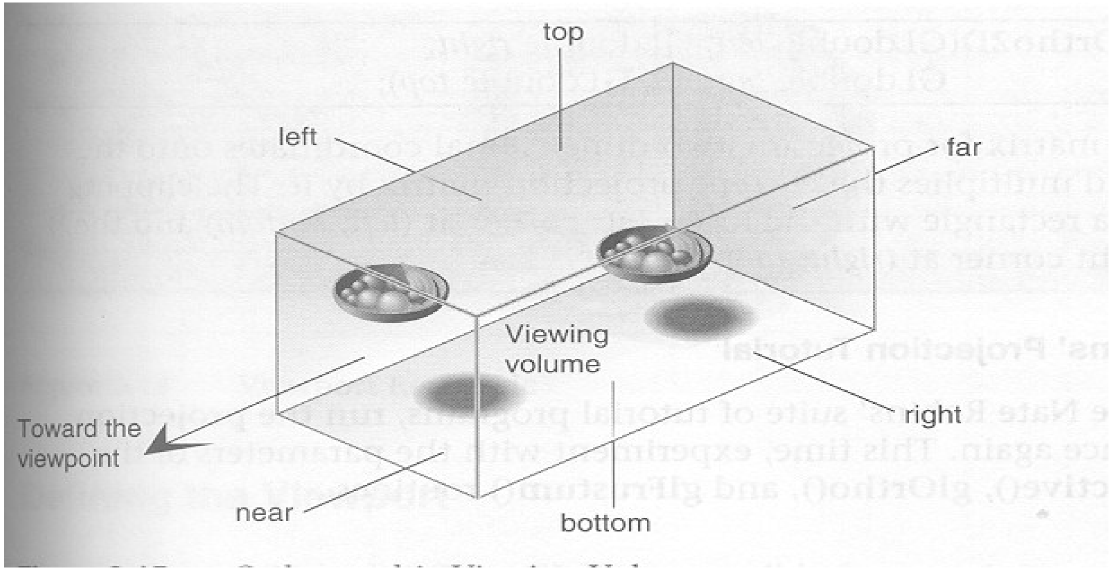
Waterfall
M.C. Escher
(1961)

OpenGL Transformation Pipeline



Parallel Projection

- Apply rotation matrix to map direction of projection to *Z* axis and up to *Y* axis
- Scale to canonical volume



From: *OpenGL Red Book*

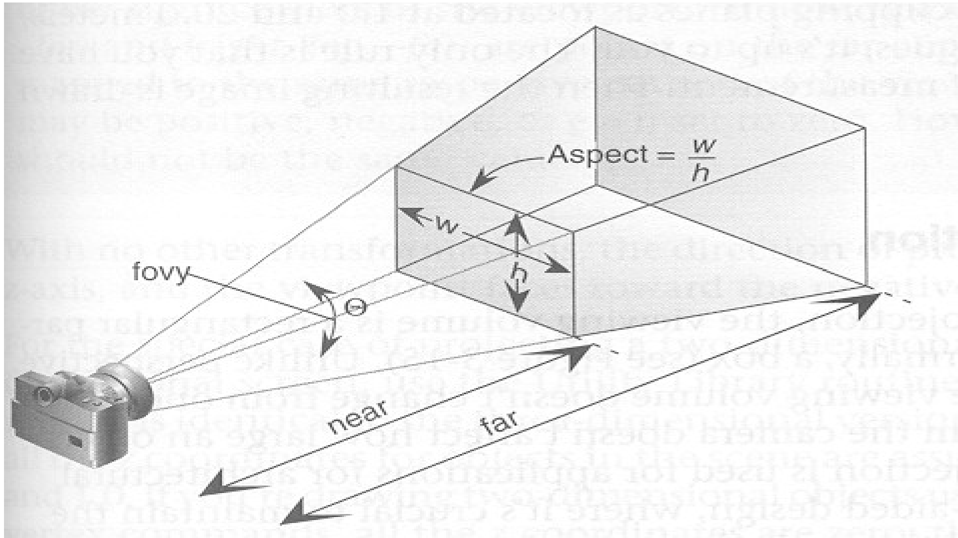
glOrtho(x_{min},x_{max},y_{min},y_{max},z_{min},z_{max})

glOrtho Projection Matrix

$$\begin{pmatrix} \frac{2}{x_{max}-x_{min}} & 0 & 0 & -\frac{x_{max}+x_{min}}{x_{max}-x_{min}} \\ 0 & \frac{2}{y_{max}-y_{min}} & 0 & -\frac{y_{max}+y_{min}}{y_{max}-y_{min}} \\ 0 & 0 & \frac{-2}{z_{max}-z_{min}} & \frac{z_{max}+z_{min}}{z_{max}-z_{min}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

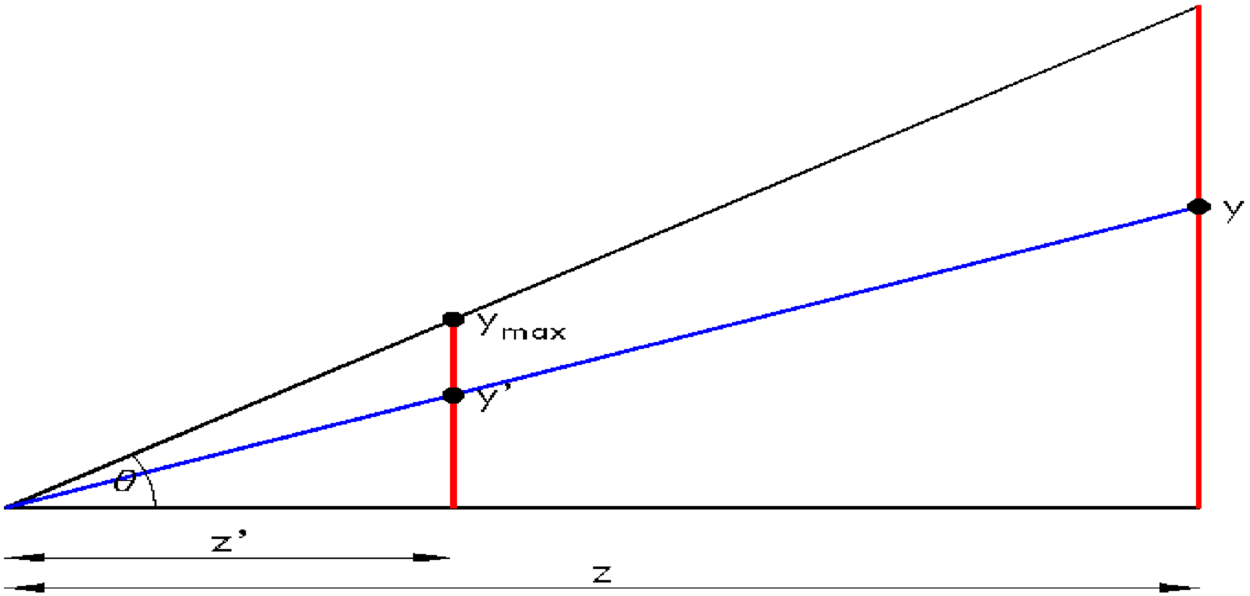
Perspective Transformation

- Apply rotation matrix to map eye position to center of scene to negative *Z* and up to *Y* axes
- Scale *(x,y)* inversely proportional to distance
- Scale to canonical volume



From: *OpenGL Red Book*

Perspective Transformation



Similar triangles: $y'/z' = y/z$, so $y' = y/z \; z'$

Let $y_{max}=1$ (NDC), $\tan\theta = y_{max} / z'$, $z' = \cot\theta$

$$\boldsymbol{y' = \cot\theta \; y/z}$$

Homogeneous Perspective Multiply

$$\begin{pmatrix} \frac{\cot\theta}{aspect} & 0 & 0 & 0 \\ 0 & \cot\theta & 0 & 0 \\ 0 & 0 & \frac{z_{far}+z_{near}}{z_{far}-z_{near}} & \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cot\theta}{aspect} \quad x \\ \cot\theta \quad y \\ \frac{z_{far}+z_{near}}{z_{far}-z_{near}} z + \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\ -z \end{pmatrix}$$

$$\equiv \begin{pmatrix} \frac{\cot\theta}{aspect} \frac{x}{z} \\ \cot\theta \frac{y}{z} \\ \frac{-2z_{far}z_{near}}{z(z_{far}-z_{near})} - \frac{z_{far}+z_{near}}{z_{far}-z_{near}} \end{pmatrix}$$

gluPerspective(fovy,aspect,Znear,Zfar)

- *fovy* is the angle in the up/down direction
- *aspect* is is the horizontal to vertical ratio
- *Znear* is the distance to the near clipping plane
  - Killer fact Znear > 0
- *Zfar* is the distance to the far clipping plane
  - *Zfar > Znear*
- *Zfar-Znear* determines *Z* resolution since the Z buffer has finite precision
  - $\log_2$(*Zfar/Znear*) bits lost

gluPerspective(fovy,aspect,Znear,Zfar)

Let θ = fovy/2

gluPerspective Projection Matrix

$$\begin{pmatrix} \frac{\cot\theta}{aspect} & 0 & 0 & 0 \\ 0 & \cot\theta & 0 & 0 \\ 0 & 0 & -\frac{z_{far}+z_{near}}{z_{far}-z_{near}} & \frac{2z_{far}z_{near}}{z_{far}-z_{near}} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

gluLookAt(E_x,E_y,E_z , C_x,C_y,C_z , U_x,U_y,U_z)

- (E_x,E_y,E_z) is the eye position
- (C_x,C_y,C_z) is the position you look at
- (U_x,U_y,U_z) is the up direction
- *C*-*E* determines the distance in the *Z* direction
- The *Z* distance to each object (from E) determines the reduction in the *(x,y)* direction

gluLookAt(E_x, E_y, E_z , C_x, C_y, C_z , U_x, U_y, U_z)

$$
\begin{array}{ll}
\text{Forward} & \mathbf{F} = \mathbf{C} - \mathbf{E} \\
\text{Sideways} & \mathbf{S} = \mathbf{F} \times \mathbf{U} \\
\text{Up} & \mathbf{U} = \mathbf{S} \times \mathbf{F}
\end{array}
$$

$$
\begin{aligned}
\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} &= \begin{pmatrix} S_x & U_x & -F_x & 0 \\ S_y & U_y & -F_y & 0 \\ S_z & U_z & -F_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -E_x \\ 0 & 1 & 0 & -E_y \\ 0 & 0 & 1 & -E_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \\
&= \begin{pmatrix} S_x & U_x & -F_x & -E_x S_x - E_y U_x + E_z F_x \\ S_y & U_y & -F_y & -E_x S_y - E_y U_y + E_z F_y \\ S_z & U_z & -F_z & -E_x S_z - E_y U_z + E_z F_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}
\end{aligned}
$$

($\mathbf{F}$ and $\mathbf{U}$ must be normalized)

First Person Navigation

- Decide where you are (E_x,E_y,E_Z)
- Decide which way you are looking
  - $(C_x,C_y,C_Z) = (E_x,E_y,E_Z) + (d_x,d_y,d_Z)$
- Decide up, e.g. (0,0,1)
- Move forward to new position
  - $(E_x,E_y,E_Z) += dt*(d_x,d_y,d_Z)$
- Turn left and right using angle
  - $(d_x,d_y,d_Z) = (\cos\theta,\sin\theta,0)$