Normals for the Twisted Cylindroid
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Consider a cylindroid where the top and bottom are both ellipses. The height of the cylinder is h along the y axis, and the major and minor axes have parameters a and b for the bottom surface and c and d for the top surface along the x and y axes, respectively. These surfaces are best described in terms of a polar coordinate θ that goes from 0 to 2π.

The rim of the bottom surface is given by

$$P_1(\theta) = \begin{pmatrix} a\cos\theta \\ -h/2 \\ b\sin\theta \end{pmatrix}$$

and the top surface by

$$P_2(\theta) = \begin{pmatrix} c\cos\theta \\ +h/2 \\ d\sin\theta \end{pmatrix}$$

The tangetial vectors along the rim of the top and bottom surfaces can be calculated by taking the derivative with respect to θ, so for the bottom surface

$$\frac{\partial P_1}{\partial\theta} = \begin{pmatrix} -a\sin\theta \\ 0 \\ b\cos\theta \end{pmatrix}$$

and for the top surface

$$\frac{\partial P_2}{\partial\theta} = \begin{pmatrix} -c\sin\theta \\ 0 \\ d\cos\theta \end{pmatrix}.$$

The vector between the corresponding points on the top and bottom surfaces are

$$P_2 - P_1 = \begin{pmatrix} (c-a)\cos\theta \\ h \\ (d-b)\sin\theta \end{pmatrix}$$

The normal vector at the rim of the bottom surface can then be calcuated as the cross product

$$\begin{aligned} N_1 &= \frac{\partial P_1}{\partial\theta} \times (P_2 - P_1) \\ &= \begin{pmatrix} -a\sin\theta \\ 0 \\ b\cos\theta \end{pmatrix} \times \begin{pmatrix} (c-a)\cos\theta \\ h \\ (d-b)\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} bh\cos\theta \\ a(b-d)\sin^2\theta + b(a-c)\cos^2\theta \\ ah\sin\theta \end{pmatrix} \end{aligned}$$

and for the top surface

$$
\begin{aligned}
N_2 &= \frac{\partial P_2}{\partial \theta} \times (P_2 - P_1) \\
&= \begin{pmatrix} -c\sin\theta \\ 0 \\ d\cos\theta \end{pmatrix} \times \begin{pmatrix} (c-a)\cos\theta \\ h \\ (d-b)\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} dh\cos\theta \\ c(b-d)\sin^2\theta + d(a-c)\cos^2\theta \\ ch\sin\theta \end{pmatrix}
\end{aligned}
$$

For a **oval cylindroid** with $a = c$ and $b = d$ the equation simplfies to

$$
N_1 = N_2 = \begin{pmatrix} b\cos\theta \\ 0 \\ a\sin\theta \end{pmatrix}
$$

For a **cylinder** with $a = b = c = d$ the equation simplfies to

$$
N_1 = N_2 = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}
$$

For a **truncated cone** with $a = b$ and $c = d$ the equation simplifies to

$$
N_1 = N_2 = \begin{pmatrix} h\cos\theta \\ a-c \\ h\sin\theta \end{pmatrix}.
$$

For a **cone** with $a = b$ and $c = d = 0$ the equation simplifies to

$$
N_1 = N_2 = \begin{pmatrix} h\cos\theta \\ a \\ h\sin\theta \end{pmatrix}.
$$