

Homogeneous Coordinates

**CSCI 4229/5229
Computer Graphics
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Basic 2D Transformations

Translation:

$$x' = x + \delta x$$

$$y' = y + \delta y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Scaling:

$$x' = S_x x$$

$$y' = S_y y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

$$x' = \cos \theta x - \sin \theta y$$

$$y' = \sin \theta x + \cos \theta y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Compound Transformations

Translation	$p' = p + d$
Scaling	$p' = Sp$
Rotation	$p' = Rp$

Sequence of transformations

$$\begin{aligned} p' &= S_3 R_3 (S_2 (R_2 R_1 S_1 (p + d_1) + d_2) + d_3) \\ &= M_3 (M_2 (M_1 (p + d_1) + d_2) + d_3) \end{aligned}$$

Homogeneous Coordinates

- Define

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \equiv \begin{pmatrix} X/W \\ Y/W \\ Z/W \end{pmatrix}$$

2D Homogeneous Transformations

Translation:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta x \\ 0 & 1 & \delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} x + w\delta x \\ y + w\delta y \\ w \end{pmatrix} \equiv \begin{pmatrix} x/w + \delta x \\ y/w + \delta y \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} S_x x \\ S_y y \\ w \end{pmatrix} \equiv \begin{pmatrix} S_x x/w \\ S_y y/w \end{pmatrix}$$

Rotation

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \cos \theta \, x - \sin \theta \, y \\ \sin \theta \, x + \cos \theta \, y \\ w \end{pmatrix} \\ &\equiv \begin{pmatrix} [\cos \theta \, x - \sin \theta \, y]/w \\ [\sin \theta \, x + \cos \theta \, y]/w \end{pmatrix} \end{aligned}$$

Compound Homogeneous Transformations

Translation	$p' = Dp$
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Scaling	$p' = Sp$
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Rotation	$p' = Rp$
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Sequence of transformations

$$\begin{aligned} p' &= S_3 R_3 D_3 S_2 D_2 R_2 R_1 S_1 D_1 p \\ &= Mp \end{aligned}$$

3D Homogeneous Translation and Scaling

Translation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta x \\ 0 & 1 & 0 & \delta y \\ 0 & 0 & 1 & \delta z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Scaling:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

3D Homogeneous Rotation

Rotation around the z axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the x axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotation around the y axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

General Homogeneous Rotation

Rotation around a unit vector (X, Y, Z) :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 + (1 - \cos \theta)(X^2 - 1) & -Z \sin \theta + (1 - \cos \theta)XY & Y \sin \theta + (1 - \cos \theta)XZ & 0 \\ Z \sin \theta + (1 - \cos \theta)XY & 1 + (1 - \cos \theta)(Y^2 - 1) & -X \sin \theta + (1 - \cos \theta)YZ & 0 \\ -Y \sin \theta + (1 - \cos \theta)XZ & X \sin \theta + (1 - \cos \theta)YZ & 1 + (1 - \cos \theta)(Z^2 - 1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Rotate x axis to unit vector (X_0, Y_0, Z_0) and y axis to unit vector (X_1, Y_1, Z_1) :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0Z_1 - Y_1Z_0 & 0 \\ Y_0 & Y_1 & Z_0X_1 - Z_1X_0 & 0 \\ Z_0 & Z_1 & X_0Y_1 - X_1Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Vector Rotation by Construction

Rotate x axis to unit vector (X_0, Y_0, Z_0) and y axis to unit vector (X_1, Y_1, Z_1) :

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & Y_0 Z_1 - Y_1 Z_0 & 0 \\ Y_0 & Y_1 & Z_0 X_1 - Z_1 X_0 & 0 \\ Z_0 & Z_1 & X_0 Y_1 - X_1 Y_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

First column by construction

$$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ w \end{pmatrix} = \begin{pmatrix} X_0 & * & * & 0 \\ Y_0 & * & * & 0 \\ Z_0 & * & * & 0 \\ 0 & * & * & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ w \end{pmatrix}$$

Second column by construction

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ w \end{pmatrix} = \begin{pmatrix} * & X_1 & * & 0 \\ * & Y_1 & * & 0 \\ * & Z_1 & * & 0 \\ * & 0 & * & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ w \end{pmatrix}$$

Rigid transformation requires third column to be the cross product.

3D Homogeneous Transformation

Transformation matrices will always be of the form

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} r_{xx} & r_{yx} & r_{zx} & t_x \\ r_{xy} & r_{yy} & r_{zy} & t_y \\ r_{xz} & r_{yz} & r_{zz} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Note that translation, scaling and rotation or any combination of these three never changes the value of w
 - There are transformations that do modify w

Order of calls in OpenGL

- What is the resulting $x' = M_1 M_2 M_3 x$ for
 - `glTranslate(1000, 2000, 3000);`
 - `glRotate(90, 0, 0, 1);`
 - `glScale(10, 20, 30);`
 - `.....`
 - `glVertex(1, 0, 0);`
- Answer: $x' = (1000, 2010, 3000)$

First Try

- Start with $(1,0,0)$
- `glTranslate(1000, 2000, 3000);`
 $(1,0,0) + (1000,2000,3000) = (1001,2000,3000)$
- `glRotate(90, 0, 0, 1);`
 $(1001,2000,3000) \rightarrow (-2000,1001,3000)$
- `glScale(10, 20, 30);`
 $(-2000,1001,3000) \rightarrow (-20000,20020,90000)$
- WRONG:
 - Correct answer: $x' = (1000,2010,3000)$

Second Try

- Start with (1,0,0)
- `glScale(10, 20, 30);`
 $(1,0,0) \rightarrow (10,0,0)$
- `glRotate(90, 0, 0, 1);`
 $(10,0,0) \rightarrow (0,10,0)$
- `glTranslate(1000, 2000, 3000);`
 $(0,10,0) + (1000,2000,3000) = (1000,2010,3000)$
- Killer fact: OpenGL ***RIGHT*** multiplies,
so read from glVertex **backwards**

OpenGL Transformation Matrices

- You can set the matrices explicitly in OpenGL
 - `glLoadMatrixd(double mat[16])`
 - `glMultMatrixd(double mat[16])`
- Setting the matrix may be convenient when you calculate vectors instead of angles
- Killer fact:
 - The matrix is stored in COLUMN major order

OpenGL 4 Transformations

- OpenGL 1 & 2 provides transformations as part of the API
- OpenGL 3 & 4 provides transformations only in the compatibility profile
 - Wants the user to generate transformations by other means
 - Provides matrix & vector operations, but user must set transformation matrices

Useful Vector Results

- **$\mathbf{c} = \mathbf{a} \times \mathbf{b}$**
 - **\mathbf{c}** is perpendicular to the plane formed by **\mathbf{a}** and **\mathbf{b}**
 - **\mathbf{c}** is zero when **\mathbf{a}** and **\mathbf{b}** are parallel
- **$\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a})$**
 - **\mathbf{c}** is perpendicular to **\mathbf{a}**
 - **\mathbf{c}** is in the plane formed by **\mathbf{a}** and **\mathbf{b}**
 - **\mathbf{c}** is zero when **\mathbf{a}** and **\mathbf{b}** are parallel