Normals for a Regular Grid

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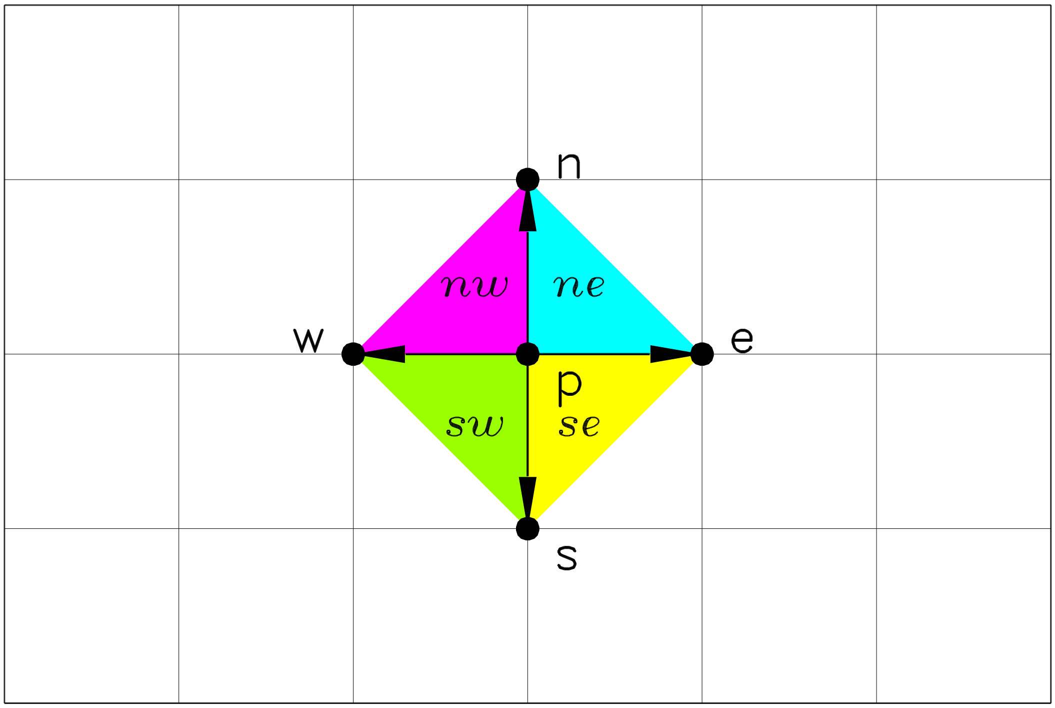

Consider a point **p** in regular grid. Let the $\vec{\mathbf{e}}$, $\vec{\mathbf{w}}$, $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ be 3D vectors from **p** to neighboring points on the grid in the cardinal directions. For a grid spacing of δ in the x and y directions and an elevation z at each point,

$$\vec{\mathbf{e}} = \left[+\delta \ , \ 0 \ , \ z_e - z_p \right]$$
$$\vec{\mathbf{w}} = \left[-\delta \ , \ 0 \ , \ z_w - z_p \right]$$
$$\vec{\mathbf{n}} = \left[\ 0 \ , \ +\delta \ , \ z_n - z_p \right]$$
$$\vec{\mathbf{s}} = \left[\ 0 \ , \ -\delta \ , \ z_s - z_p \right]$$

Gouraud averaging require that we calculate the average of the normal vectors of triangles that meet at **p**. For a regular grid, there are four triangles in the north-east, north-west, south-west and south-east quadrants relative to **p**, respectively.

The normal for each of these triangles can be calculated as

$$\vec{\mathbf{N}}_{ne} = \vec{\mathbf{e}} \times \vec{\mathbf{n}}$$
$$\vec{\mathbf{N}}_{nw} = \vec{\mathbf{n}} \times \vec{\mathbf{w}}$$
$$\vec{\mathbf{N}}_{sw} = \vec{\mathbf{w}} \times \vec{\mathbf{s}}$$
$$\vec{\mathbf{N}}_{se} = \vec{\mathbf{s}} \times \vec{\mathbf{e}}$$

When $\delta \gg \delta z$, then $\vec{\mathbf{e}}$, $\vec{\mathbf{w}}$, $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ are of approximately equal magnitude and we can simply add them together before normalizing the resulting sum.

Using the anticommutative property of cross products

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

and the distributive property of cross products

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}} = \vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}})$$

we get

$$\begin{aligned}
\vec{\mathbf{N}}_{ne} + \vec{\mathbf{N}}_{se} + \vec{\mathbf{N}}_{nw} + \vec{\mathbf{N}}_{sw} &= \vec{\mathbf{e}} \times \vec{\mathbf{n}} \; + \; \vec{\mathbf{s}} \times \vec{\mathbf{e}} \; + \; \vec{\mathbf{n}} \times \vec{\mathbf{w}} \; + \; \vec{\mathbf{w}} \times \vec{\mathbf{s}} \\
&= \vec{\mathbf{e}} \times \vec{\mathbf{n}} \; - \; \vec{\mathbf{e}} \times \vec{\mathbf{s}} \; - \; \vec{\mathbf{w}} \times \vec{\mathbf{n}} \; + \; \vec{\mathbf{w}} \times \vec{\mathbf{s}} \\
&= \vec{\mathbf{e}} \times (\vec{\mathbf{n}} - \vec{\mathbf{s}}) \; - \; \vec{\mathbf{w}} \times (\vec{\mathbf{n}} - \vec{\mathbf{s}}) \\
&= (\vec{\mathbf{e}} - \vec{\mathbf{w}}) \times (\vec{\mathbf{n}} - \vec{\mathbf{s}}) \\
&= \big[2\delta \; , \; 0 \; , \; z_e - z_w \big] \times \big[\; 0 \; , \; 2\delta \; , \; z_n - z_s \big] \\
&= \big[2\delta (z_w - z_e) \; , \; 2\delta (z_s - z_n) \; , \; 4\delta^2 \big]
\end{aligned}$$

The average normal vector at the point $\mathbf{p}$ is therefore

$$\vec{\mathbf{N}}_p \approx \frac{\vec{\mathbf{N}}_{ne} + \vec{\mathbf{N}}_{se} + \vec{\mathbf{N}}_{nw} + \vec{\mathbf{N}}_{sw}}{4\delta} = \big[\tfrac{1}{2}(z_w - z_e) \; , \; \tfrac{1}{2}(z_s - z_n) \; , \; \delta \big]$$

This approximation is often good enough for procedurally generated terrain.